Asymptotic analysis of an advection-diffusion equation and application to boundary controllability

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Abstract

We perform the asymptotic analysis of the scalar advection-diffusion equation $y^\varepsilon_t - \varepsilon y^\varepsilon_{xx} + M y^\varepsilon_x = 0$, $(x,t) \in (0,1) \times (0,T)$, with respect to the diffusion coefficient ε. We use the matched asymptotic expansion method which allows to describe the boundary layers of the solution. We then use the asymptotics to discuss the controllability property of the solution for $T \geq 1/M$.

Key words: Asymptotic analysis, Boundary layers, Singular controllability.

1 Introduction - Problem statement

Let $L > 0$, $T > 0$ and $Q_T := (0,L) \times (0,T)$. This work is concerned with the scalar advection-diffusion equation

$$\begin{cases} y^\varepsilon_t - \varepsilon y^\varepsilon_{xx} + M y^\varepsilon_x = 0 & \text{in} \quad Q_T, \\ y^\varepsilon(0,\cdot) = v^\varepsilon,\ y^\varepsilon(L,\cdot) = 0 & \text{on} \quad (0,T), \\ y^\varepsilon(\cdot,0) = y^\varepsilon_0 & \text{in} \quad (0,L), \end{cases} \tag{1}$$

where $y^\varepsilon_0 \in H^{-1}(0,L)$ is the initial data. $\varepsilon > 0$ is the diffusion coefficient while $M \in \mathbb{R}^\star$ is the transport coefficient; $v^\varepsilon = v^\varepsilon(t)$ is the control function in $L^2(0,T)$ and $y^\varepsilon = y^\varepsilon(x,t)$ is the associated state.

For any $y^\varepsilon_0 \in H^{-1}(0,L)$ and $v^\varepsilon \in L^2(0,T)$, there exists exactly one solution y^ε to (1), with the regularity $y^\varepsilon \in L^2(Q_T) \cap C([0,T]; H^{-1}(0,L))$. Moreover, as $\varepsilon \to 0^+$, the system (1) "degenerates" into a transport equation: precisely, assuming that $v^\varepsilon \rightharpoonup v$ in $L^2(0,T)$ and that the initial data y^ε_0 is independent of ε, then the solution y^ε of (1) weakly converges in $L^2(Q_T)$ towards y solution of the equation

$$\begin{cases} y_t + M y_x = 0 & \text{in} \quad Q_T, \\ y(0,\cdot) = v & \text{on} \quad (0,T) \quad \text{if} \quad M > 0, \\ y(L,\cdot) = 0 & \text{on} \quad (0,T) \quad \text{if} \quad M < 0, \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{2}$$

We refer to [2], Proposition 1.

We are interested in this work with a precise asymptotic description of the solution y^ε when ε is small. As a first motivation, we can mention that system (1) can be seen as a simple example of complex models where the diffusion coefficient is very small compared to the others. We have notably in mind the Stokes system where ε stands for the viscosity coefficient. A second motivation comes from the asymptotic controllability property of (1) recently studied in [2, 4, 9, 10] and which exhibits some apparently surprising behaviors. More precisely, for any final time $T > 0$, $\varepsilon > 0$ and $y^\varepsilon_0 \in H^{-1}(0,L)$, there

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exist control functions $v^\varepsilon \in L^2(0,T)$ such that the corresponding solution to (1) satisfies $y^\varepsilon(\cdot,T)=0$ in $H^{-1}(0,L)$ (see [3, 6]). This raises the question of the asymptotic behavior as $\varepsilon \to 0$ of the cost of control defined by

$$K(\varepsilon,T,M) := \sup_{\|y_0\|_{L^2(0,L)}=1} \left\{ \min_{u\in\mathcal{C}(y_0,T,\varepsilon,M)} \|u\|_{L^2(0,T)} \right\}, \tag{3}$$

where $\mathcal{C}$ denotes the (non-empty) set of null controls

$$\mathcal{C}(y_0,T,\varepsilon,M) := \left\{ v\in L^2(0,T); y=y(v) \text{ solves (1) and satisfies } y^\varepsilon(\cdot,T)=0 \,\text{in}\, H^{-1}(0,L) \right\}.$$

The minimal time T_M for which this cost is uniformly bounded with respect to ε is unknown. It is proved in [2] that $T_M \in [1,4.3]L/M$ for $M>0$ and $T_M\in[2,57.2]L/|M|$ for $M<0$. Precisely, if $T<L/M$ (resp. $T<2L/|M|$) for $M>0$ (resp. $M<0$), then the cost $K(\varepsilon,T,M)$ blows up exponentially as $\varepsilon\to 0^+$: such behavior is achieved with the following initial condition

$$y_0^\varepsilon(x) = K_\varepsilon e^{-\frac{Mx}{2\varepsilon}} \sin\left(\frac{\pi x}{L}\right), \quad K_\varepsilon = \left(\frac{2\pi^2\varepsilon^3(1-e^{-\frac{LM}{\varepsilon}})}{M(M^2L^2+4\pi^2\varepsilon^2)} \right)^{-1/2} = \mathcal{O}(\varepsilon^{-3/2}), \tag{4}$$

so that $\|y_0^\varepsilon\|_{L^2(0,L)}=1$. This data get concentrated at $x=0$ (resp. $x=L$) for $M>0$ (resp. $M<0$). The bounds for T_M have then been improved in [4] and in [10] successively. These bounds for $M<0$ are apparently not expected since, first the cost of control $K(0,T,M)$ for (2) is zero as soon as $T\geq L/|M|$ and second, because we can check that the L^2-norm of y^ε, solution of (1) with $v^\varepsilon\equiv 0$ satisfies the inequality

$$\|y^\varepsilon(\cdot,t)\|_{L^2(0,L)} \leq \|y^\varepsilon(\cdot,0)\|_{L^2(0,L)} e^{-c\,\varepsilon t}, \quad \forall t > \frac{L}{|M|} \tag{5}$$

for some constant $c>0$ independent of ε. In other words, the null function $v^\varepsilon\equiv 0$ is an approximate null control for (1) for $T>L/|M|$. One may then conclude that the controllability property for (1) and the limit as $\varepsilon\to 0^+$ do not commute. However, it should be noted that the initial condition (4) does not fall in the framework of the weak convergence result stated above as it depends on ε ! Nevertheless, the time T_M and more generally the behavior of the control of minimal L^2-norm (which appears in (3)) remains unclear for ε small: there is a kind of balance between the term $-\varepsilon y^\varepsilon_{xx}$ which favor the diffusion (and so the null controllability) for ε large and the term My^ε_x which enhance the complete transport of the solution out of the domain $(0,L)$ for ε small.

One may tackle this problem and the determination of the minimal uniform controllability time T_M by numerical methods: this consists in approximating the cost $K(\varepsilon,T,M)$ for various values of ε and $T>0$, the ratio L/M being fixed. This has been done in [12] for ε in the range $[10^{-3},10^{-1}]$ and suggests that for $M>0$, T_M is equal to L/M achieved with initial conditions concentrating at $x=0$ closed to (4). The case $M<0$ for which the transport term acts "against" the control is much more involved, the underlying approximated problem being highly ill-conditioned. Smaller values of ε are difficult to consider numerically: in view of (5), the norm $\|y^\varepsilon(\cdot,t)\|_{L^2(0,L)}$ decreases very fast under the zero "numeric", which is of the order $\mathcal{O}(10^{-16})$ when the double digit precision is used.

An alternative theoretical approach is to analyze, through an asymptotic analysis with respect to the parameter ε, the structure of the (unique) control of minimal L^2-norm, the initial condition y_0^ε being fixed. In this respect, we may use the fact that such control is characterized by the following optimality system

$$\begin{cases} y^\varepsilon_t - \varepsilon y^\varepsilon_{xx} + My^\varepsilon_x = 0, \quad -\varphi^\varepsilon_t - \varepsilon y^\varepsilon_{xx} - M\varphi^\varepsilon_x = 0, & (x,t)\in Q_T, \\ y^\varepsilon(\cdot,0)=y_0^\varepsilon, \quad y^\varepsilon(\cdot,T)=0, & x\in(0,L), \\ v^\varepsilon(t) = y^\varepsilon(0,t) = \varepsilon\varphi^\varepsilon_x(0,t), & t\in(0,T), \\ y^\varepsilon(L,t) = \varphi^\varepsilon(0,t) = \varphi^\varepsilon(L,t) = 0, & t\in(0,T), \end{cases} \tag{6}$$

φ^ε being the adjoint solution. We are then faced to the asymptotic analysis of a partial differential system with respect to a small parameter, in a spirit for instance of the book [7] in the closed context of optimal control theory. In view of (5), such asymptotic analysis should be as precise as possible in order to fill the gap between the approximate null controllability achieved with $v^\varepsilon \equiv 0$ and the null controllability leading to exponentially large controls. However, in spite of the apparent simplicity of the system (1), such analysis is not straightforward because, as ε goes to zero, the direct and adjoint solutions exhibit boundary layers in the transition parabolic-hyperbolic. For example, for $M > 0$, in agreement with the structure of the weak limit (2), the solution y^ε (resp. φ^ε) exhibits a first boundary layer of size $\mathcal{O}(\varepsilon)$ at $x = L$ (resp. $x = 0$). Moreover, the solution y^ε (resp. φ^ε) exhibits a second boundary layer of size $\mathcal{O}(\sqrt{\varepsilon})$ along the characteristic $\{(x,t) \in Q_T, Lx - Mt = 0\}$ (resp. $\{(x,t) \in Q_T, Lx - M(t-T) - 1 = 0\}$). A third singular behavior due to the initial condition y_0^ε occurs for y^ε in the neighborhood of the points $(x_0,t_0) = (0,0)$ and $(x_1,t_1) = (L,0)$.

Remark however that the boundary layer for y^ε on the characteristic does not occur if and only if the function v^ε and the initial condition y_0^ε satisfy some compatibility conditions at the point (x_0,t_0). Remark also that the optimal control v^ε, supported on $\{0\} \times (0,T)$, lives in the first boundary layer for φ^ε.

Similar boundary layers occur for $M < 0$.

The main purpose of this work, devoted to the case $M > 0$, is to perform an asymptotic analysis of the direct problem (1), assuming v^ε fixed but satisfying appropriate compatibility conditions at the initial $t = 0$ with the initial condition y_0^ε as $x = 0$. We therefore focus on the boundary layers appearing at $x = L$, employing the matched asymptotic expansion method described in the book of M. VanDyke, see [15].

This work is organized as follows. In Section 2, for a fixed function v^ε, we perform the asymptotic analysis of the direct problem (1). Precisely, assuming that the initial condition is independent of ε and that the control function v^ε is given by $v^\varepsilon = \sum_{k=0}^m \varepsilon^k v^k$, we construct an approximation w_m^ε of the solution y^ε. The matched asymptotic expansion method is used in section 2.1 to define an outer solution (out of the boundary layer) and an inner solution. Upon regularity assumptions on the functions v^k, $k = 0, \dots, m$ and y_0^ε, plus compatibility conditions between the derivatives of the functions v^k and the derivatives of y_0^ε at (x_0,t_0), we prove that w_m^ε is a regular and strong convergent approximation of y^ε, as $\varepsilon \to 0^+$. The estimate between w_m^ε and y^ε involves the initial boundary layer function, exponentially small with respect to ε (see Lemma 2.8). The analysis is done in the case $m = 2$ in section 2.2 (see Theorem 2.1) and in the general case in section 2.3 (see Theorem 2.2). In Theorem 2.3, we then provide sufficient conditions on the control functions v^k and on y_0 allowing to pass to the limit as $m \to \infty$ with ε small enough but fixed. A similar analysis is conducted for the adjoint solution φ^ε in section 2.5. We then use such asymptotic to deduce in Section 3 some approximate controllability results for $T \geq L/M$. In Section 4, we discuss the case of initial conditions which depend on ε, in particular the one defined by (4). The final section 5 discusses the limits of such asymptotic analysis to discuss the system (6) and present some perspectives.

As far as we know, there are few works in the literature dealing both with asymptotic analysis and controllability. The chapter 3 of [8] entitled "Exact controllability and singular perturbation" studies the controllability property of the equation $y'' + \varepsilon\Delta^2 y - \Delta y = 0$ as $\varepsilon \to 0^+$ and identifies the limit control problem. We mention the recent work [11] where the controllability of a Burgers equation $y_t - y_{xx} + yy_x = 0$ in small time is discussed, leading after a change of variable to a small parameter in front of the linear second order term. We also mention [13] where a vanishing viscosity method is employed to study the sensitivity of an optimal control problem.

In the sequel, we shall use the following notations:

$$L_\varepsilon(y) := y_t - \varepsilon y_{xx} + My_x, \quad L_\varepsilon^\star(\varphi) := -\varphi_t - \varepsilon\varphi_{xx} - M\varphi_x.$$

Without loss of generality, we assume henceforth that $L = 1$.

2 Matched asymptotic expansions and approximate solutions

In this section we consider the solution of the problem (1). We apply the method of matched asymptotic expansions to construct approximate solutions. We refer to [5, 14, 15] for a general presentation of the method. Then we apply the same procedure to construct asymptotic approximate solutions of the adjoint solution φ^ε, see problem (6).

Let us consider the problem

$$\begin{cases} y^\varepsilon_t - \varepsilon y^\varepsilon_{xx} + M y^\varepsilon_x = 0, & (x,t) \in Q_T, \\ y^\varepsilon(0,t) = v^\varepsilon(t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{7}$$

where y_0 and v^ε are given functions. We assume that $M > 0$ and v^ε is in the form $v^\varepsilon = \sum_{k=0}^{m} \varepsilon^k v^k$, the functions $v^0,\ v^1, \cdots, v^m$ being known. We construct an asymptotic approximation of the solution y^ε of (7) by using the method of matched asymptotic expansions. We assume here that the initial condition $y^\varepsilon(x,0)$ is independent of ε but the procedure is very similar for $y^\varepsilon(\cdot,0)$ of the form $y^\varepsilon(\cdot,0) = \sum_{k=0}^m \varepsilon^k y_0^k$. The case $M < 0$ can be treated similarly.

In the sequel, $c,\ c_1,\ c_2, \cdots$, will stand for generic constants that do not depend on ε. When the constants $c,\ c_1,\ c_2, \cdots$, depend in addition on some other parameter p we will write $c_p,\ c_1(p),\ c_2(p), \cdots$

2.1 Formal asymptotic expansions

Let us consider two formal asymptotic expansions of y^ε:
– the outer expansion

$$\sum_{k=0}^{m} \varepsilon^k y^k(x,t), \quad (x,t) \in Q_T,$$

– the inner expansion

$$\sum_{k=0}^{m} \varepsilon^k Y^k(z,t), \quad z = \frac{1-x}{\varepsilon} \in (0, \varepsilon^{-1}),\ t \in (0,T).$$

We will construct outer and inner expansions which will be valid in the so-called outer and inner regions, respectively. Here the boundary layer (inner region) occurs near $x = 1$, it is of $\mathcal{O}(\varepsilon)$ size, and the outer region is the subset of $(0,1)$ consisting of the points far from the boundary layer, it is of $\mathcal{O}(1)$ size. There is an intermediate region between them, with size $\mathcal{O}(\varepsilon^\gamma)$, $\gamma \in (0,1)$. To construct an approximate solution we require that inner and outer expansions coincide in the intermediate region, then some conditions must be satisfied in that region by the inner and outer expansions. These conditions are the so-called matching asymptotic conditions.

Putting $\sum_{k=0}^{m} \varepsilon^k y^k(x,t)$ into equation $(7)_1$, the identification of the powers of ε yields

$$\begin{aligned} &\varepsilon^0: \quad y^0_t + M y^0_x = 0, \\ &\varepsilon^k: \quad y^k_t + M y^k_x = y^{k-1}_{xx}, \quad \text{for any } 1 \le k \le m. \end{aligned}$$

Taking the initial and boundary conditions into account we define y^0 and y^k $(1 \le k \le m)$ as functions satisfying the transport equations, respectively,

$$\begin{cases} y^0_t + M y^0_x = 0, & (x,t) \in Q_T, \\ y^0(0,t) = v^0(t), & t \in (0,T), \\ y^0(x,0) = y_0(x), & x \in (0,1), \end{cases} \tag{8}$$

and

$$\begin{cases} y^k_t + My^k_x = y^{k-1}_{xx}, & (x,t) \in Q_T, \\ y^k(0,t) = v^k(t), & t \in (0,T), \\ y^k(x,0) = 0, & x \in (0,1). \end{cases} \tag{9}$$

The solution of (8) is given by

$$y^0(x,t) = \begin{cases} y_0(x - Mt), & x > Mt, \\ v^0\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases} \tag{10}$$

Using the method of characteristics we find that, for any $1 \le k \le m$,

$$y^k(x,t) = \begin{cases} \displaystyle\int_0^t y^{k-1}_{xx}(x + (s-t)M, s)ds, & x > Mt, \\ v^k\left(t - \dfrac{x}{M}\right) + \displaystyle\int_0^{x/M} y^{k-1}_{xx}(sM, t - \frac{x}{M} + s)ds, & x < Mt. \end{cases} \tag{11}$$

Remark 1 *We actually verify that we have explicitly*

$$y^1(x,t) = \begin{cases} t\, y_0^{(2)}(x - Mt), & x > Mt, \\ v^1\left(t - \dfrac{x}{M}\right) + \dfrac{x}{M^3}(v^0)^{(2)}\left(t - \dfrac{x}{M}\right), & x < Mt, \end{cases} \tag{12}$$

and

$$y^2(x,t) = \begin{cases} \dfrac{t^2}{2}\, y_0^{(4)}(x - Mt), & x > Mt, \\ v^2\left(t - \dfrac{x}{M}\right) + \dfrac{x}{M^3}(v^1)^{(2)}\left(t - \dfrac{x}{M}\right) & \\ \quad - \dfrac{2x}{M^5}(v^0)^{(3)}\left(t - \dfrac{x}{M}\right) + \dfrac{x^2}{2M^6}(v^0)^{(4)}\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases} \tag{13}$$

Here and in the sequel, $f^{(i)}$ denotes the derivative of order i of the real function f.

Now we turn back to the construction of the inner expansion. Putting $\sum_{k=0}^{m} \varepsilon^k Y^k(z,t)$ into equation $(7)_1$, the identification of the powers of ε yields

$$\begin{aligned} \varepsilon^{-1}: &\quad Y^0_{zz}(z,t) + MY^0_z(z,t) = 0, \\ \varepsilon^{k-1}: &\quad Y^k_{zz}(z,t) + MY^k_z(z,t) = Y^{k-1}_t(z,t), \qquad \text{for any } 1 \le k \le m. \end{aligned}$$

We impose that $Y^k(0,t) = 0$ for any $0 \le k \le m$. To get the asymptotic matching conditions we write that, for any fixed t and large z,

$$\begin{aligned} &Y^0(z,t) + \varepsilon Y^1(z,t) + \varepsilon^2 Y^2(z,t) + \cdots + \varepsilon^m Y^m(z,t) \\ &= y^0(x,t) + \varepsilon y^1(x,t) + \varepsilon^2 y^2(x,t) + \cdots + \varepsilon^m y^m(x,t) + \mathcal{O}(\varepsilon^{m+1}). \end{aligned}$$

Rewriting the right-hand side of the above equality in terms of z, t and using Taylor expansions we have

$$\begin{aligned} &Y^0(z,t) + \varepsilon Y^1(z,t) + \varepsilon^2 Y^2(z,t) + \cdots + \varepsilon^m Y^m(z,t) \\ &= y^0(1 - \varepsilon z, t) + \varepsilon y^1(1 - \varepsilon z, t) + \varepsilon^2 y^2(1 - \varepsilon z, t) + \cdots + \varepsilon^m y^m(1 - \varepsilon z, t) + \mathcal{O}(\varepsilon^{m+1}) \\ &= y^0(1,t) + y^0_x(1,t)(-\varepsilon z) + \frac{1}{2} y^0_{xx}(1,t)(\varepsilon z)^2 + \cdots + \frac{1}{m!}(y^0)_x^{(m)}(1,t)(-\varepsilon z)^m \\ &\quad + \varepsilon\left(y^1(1,t) + y^1_x(1,t)(-\varepsilon z) + \frac{1}{2} y^1_{xx}(1,t)(\varepsilon z)^2 + \cdots + \frac{1}{(m-1)!}(y^0)_x^{(m-1)}(1,t)(-\varepsilon z)^{m-1}\right) \\ &\quad + \cdots\cdots + \varepsilon^m y^m(1,t) + \mathcal{O}(\varepsilon^{m+1}). \end{aligned}$$

Therefore the matching conditions read

$$
\begin{aligned}
&Y^0(z,t) \sim Q^0(z,t) := y^0(1,t), \quad \text{as } z \to +\infty,\\
&Y^1(z,t) \sim Q^1(z,t) := y^1(1,t) - y^0_x(1,t)z, \quad \text{as } z \to +\infty,\\
&Y^2(z,t) \sim Q^2(z,t) := y^2(1,t) - y^1_x(1,t)z + \frac{1}{2}y^0_{xx}(1,t)z^2, \quad \text{as } z \to +\infty,\\
&\dots\\
&Y^m(z,t) \sim Q^m(z,t) := y^m(1,t) - y^{m-1}_x(1,t)z + \frac{1}{2}y^{m-2}_{xx}(1,t)z^2 + \dots + \frac{1}{m!}(y^0)^{(m)}_x(1,t)(-z)^m,\\
&\text{as } z \to +\infty.
\end{aligned}
$$

We thus define Y^0 as a solution of

$$
\begin{cases}
Y^0_{zz}(z,t) + MY^0_z(z,t) = 0, & (z,t) \in (0,+\infty) \times (0,T),\\
Y^0(0,t) = 0, & t \in (0,T),\\
\lim\limits_{z\to+\infty} Y^0(z,t) = \lim\limits_{x\to 1} y^0(x,t), & t \in (0,T).
\end{cases}
\tag{14}
$$

The last condition in (14) is the matching asymptotic condition. The general solution of $(14)_1$, $(14)_2$ is

$$
Y^0(z,t) = C(t)\left(1 - e^{-Mz}\right),
$$

where $C(t)$ is an arbitrary constant. The matching condition allows to find $C(t) = y^0(1,t)$, therefore the solution of (14) is

$$
Y^0(z,t) = y^0(1,t)\left(1 - e^{-Mz}\right), \quad (z,t) \in (0,+\infty) \times (0,T). \tag{15}
$$

Next we determine the general solution of

$$
\begin{aligned}
&Y^1_{zz}(z,t) + MY^1_z(z,t) = y^0_t(1,t)\left(1 - e^{-Mz}\right), && (z,t) \in (0,+\infty) \times (0,T),\\
&Y^1(0,t) = 0, && t \in (0,T).
\end{aligned}
$$

We find

$$
Y^1(z,t) = \left(C(t) - y^0_x(1,t)z\right) + e^{-Mz}\left(-C(t) - y^0_x(1,t)z\right),
$$

where $C(t)$ is an arbitrary constant. We determine $C(t)$ by using the matching asymptotic condition

$$
\lim_{z\to+\infty}[Y^1(z,t) - Q^1(z,t] = 0, \quad t \in (0,T),
$$

which gives

$$
Y^1(z,t) = \left(y^1(1,t) - y^0_x(1,t)z\right) + e^{-Mz}\left(-y^1(1,t) - y^0_x(1,t)z\right). \tag{16}
$$

The function Y^2 is defined as a solution of

$$
\begin{cases}
Y^2_{zz}(z,t) + MY^2_z(z,t) = Y^1_t(z,t), & (z,t) \in (0,+\infty) \times (0,T),\\
Y^2(0,t) = 0, & t \in (0,T),\\
\lim\limits_{z\to+\infty}[Y^2(z,t) - Q^2(z,t] = 0, & t \in (0,T).
\end{cases}
$$

We obtain

$$
Y^2(z,t) = \left(y^2(1,t) - y^1_x(1,t)z + y^0_{xx}(1,t)\frac{z^2}{2}\right) + e^{-Mz}\left(-y^2(1,t) - y^1_x(1,t)z - y^0_{xx}(1,t)\frac{z^2}{2}\right). \tag{17}
$$

For $1 \le k \le m$, the function Y^k is defined iteratively as the solution of

$$
\begin{cases}
Y^k_{zz}(z,t) + MY^k_z(z,t) = Y^{k-1}_t(z,t), & (z,t) \in (0,+\infty) \times (0,T),\\
Y^k(0,t) = 0, & t \in (0,T),\\
\lim\limits_{z\to+\infty}[Y^k(z,t) - Q^k(z,t)] = 0, & t \in (0,T).
\end{cases}
\tag{18}
$$

2.2 Second order approximation

Here we take $m = 2$. The outer expansion is $\sum_{k=0}^{2} \varepsilon^k y^k(x,t)$, where y^0 and y^k $(k = 1, 2)$ are given by (10) and (11), respectively, and the inner expansion is $\sum_{k=0}^{2} \varepsilon^k Y^k(z,t)$, where Y^0, Y^1 and Y^2 are given by (15), (16) and (17), respectively. We introduce a C^∞ cut-off function $\mathcal{X} : \mathbb{R} \to [0,1]$ such that

$$\mathcal{X}(s) = \begin{cases} 1, & s \geq 2, \\ 0, & s \leq 1, \end{cases} \tag{19}$$

and define, for $\gamma \in (0,1)$, the function $\mathcal{X}_\varepsilon : [0,1] \to [0,1]$, plotted on Figure 1, by

$$\mathcal{X}_\varepsilon(x) = \mathcal{X}\left(\frac{1-x}{\varepsilon^\gamma}\right). \tag{20}$$

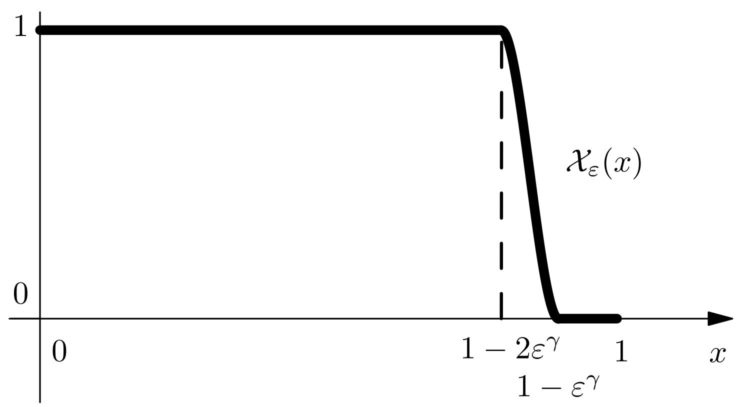

Figure 1: The function $\mathcal{X}_\varepsilon$.

Then we introduce the function w_2^ε by

$$w_2^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^{2} \varepsilon^k y^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{2} \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right), \quad (x,t) \in Q_T, \tag{21}$$

defined to be the second order asymptotic approximation of the solution y^ε of (7). To justify all the computations we will perform we need some regularity assumptions on the data y_0, v^0, v^1 and v^2. We have the following result.

LEMMA 2.1 *(i) Assume that $y_0 \in C^5[0,1]$, $v^0 \in C^5[0,T]$ and the following C^5-matching conditions are satisfied*

$$M^p(y_0)^{(p)}(0) + (-1)^{p+1}(v^0)^{(p)}(0) = 0, \quad 0 \leq p \leq 5. \tag{22}$$

Then the function y^0 defined by (10) belongs to $C^5(\overline{Q_T})$.
(ii) Additionally, assume that $v^1 \in C^3[0,T]$, $v^2 \in C^1[0,T]$ and the following C^3 and C^1-matching conditions are satisfied, respectively,

$$\begin{cases} v^1(0) = 0, \quad (v^1)^{(1)}(0) = M^{-2}(v^0)^{(2)}(0) = y_0^{(2)}(0), \\ (v^1)^{(2)}(0) = 2M^{-2}(v^0)^{(3)}(0) = -2My_0^{(3)}(0), \\ (v^1)^{(3)}(0) = 3M^{-2}(v^0)^{(4)}(0) = 3M^2 y_0^{(4)}(0), \end{cases} \tag{23}$$

$$v^2(0) = 0, \quad (v^2)^{(1)}(0) = 0. \tag{24}$$

Then the function y^1 defined by (11) *(with $k = 1$) belongs to $C^3(\overline{Q_T})$, and the function y^2 defined by* (11) *(with $k = 2$) belongs to $C^1(\overline{Q_T})$.*

PROOF. *(i)* According to the explicit form (10), it suffices to match the partial derivative of y^0 on the characteristic line $\{(x,t), x - Mt = 0\}$. Differentiating (10) p times ($p \leq 5$) with respect to x we have

$$\frac{\partial^p y^0}{\partial x^p}(x,t) = \begin{cases} y_0^{(p)}(x - Mt), & x > Mt, \\ \dfrac{(-1)^p}{M^p}(v^0)^{(p)}\left(t - \dfrac{x}{M}\right), & x < Mt. \end{cases}$$

Matching the expressions of $\dfrac{\partial^p y^0}{\partial x^p}$ upper and under the characteristic line $\{(x,t), x - Mt = 0\}$ gives (22) and ensures the continuity of $\dfrac{\partial^p y^0}{\partial x^p}$ in $\overline{Q_T}$. Differentiating (10) p times with respect to t we have

$$\frac{\partial^p y^0}{\partial t^p}(x,t) = \begin{cases} (-1)^p M^p y_0^{(p)}(x - Mt), & x > Mt, \\ (v^0)^{(p)}\left(t - \dfrac{x}{M}\right), & x < Mt, \end{cases}$$

then we see that the continuity of $\dfrac{\partial^p y^0}{\partial t^p}$ holds under condition (22). Using equation (8) we easily verify that the mixed partial derivatives, of order $p \leq 5$, of y^0 are continuous under condition (22).
(ii) Arguing as previously, using formula (12) and equation (9) (with $k = 1$) we find the matching conditions (23). Then, using formula (13) and equation (9) (with $k = 2$) we find the matching conditions (24). □

We have the following result.

LEMMA 2.2 *Let w_2^ε be the function defined by* (21). *Assume that the assumptions of Lemma 2.1 hold true. Then there is a constant c independent of ε such that*

$$\|L_\varepsilon(w_2^\varepsilon)\|_{C([0,T];L^2(0,1))} \leq c\varepsilon^{\frac{5\gamma}{2}}. \tag{25}$$

PROOF. A straightforward calculation gives

$$L_\varepsilon(w_2^\varepsilon)(x,t) = \sum_{i=1}^{5} I_\varepsilon^i(x,t),$$

with

$$\begin{aligned}
I_\varepsilon^1(x,t) &= -\varepsilon^3 y_{xx}^2(x,t)\mathcal{X}_\varepsilon(x), \\
I_\varepsilon^2(x,t) &= \varepsilon^2(1 - \mathcal{X}_\varepsilon(x))Y_t^2\left(\frac{1-x}{\varepsilon}, t\right), \\
I_\varepsilon^3(x,t) &= M\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{-\gamma}\left(\sum_{k=0}^{2}\varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right) - \sum_{k=0}^{2}\varepsilon^k y^k(x,t)\right), \\
I_\varepsilon^4(x,t) &= \mathcal{X}''\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-2\gamma}\left(\sum_{k=0}^{2}\varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right) - \sum_{k=0}^{2}\varepsilon^k y^k(x,t)\right), \\
I_\varepsilon^5(x,t) &= 2\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-\gamma}\left(\varepsilon^{-1}\sum_{k=0}^{2}\varepsilon^k Y_z^k\left(\frac{1-x}{\varepsilon}, t\right) + \sum_{k=0}^{2}\varepsilon^k y_x^k(x,t)\right).
\end{aligned}$$

Clearly,

$$\|I_\varepsilon^1\|_{C([0,T];L^2(0,1))} \le \varepsilon^3 \|y_{xx}^2\|_{C([0,T];L^2(0,1))} \le c\varepsilon^3, \tag{26}$$

and

$$\begin{aligned}\|I_\varepsilon^2\|_{C([0,T];L^2(0,1))} &\le \varepsilon^2 \left\| (1-\mathcal{X}_\varepsilon(x)) Y_t^2\left(\frac{1-x}{\varepsilon}, t\right)\right\|_{C([0,T];L^2(0,1))} \\ &\le \varepsilon^2 \max_{t\in[0,T]} \left(\int_{1-2\varepsilon^\gamma}^1 \left| Y_t^2\left(\frac{1-x}{\varepsilon}, t\right)\right|^2 dx\right)^{1/2}.\end{aligned}$$

Using a change of variable we have

$$\left(\int_{1-2\varepsilon^\gamma}^1 \left| Y_t^2\left(\frac{1-x}{\varepsilon}, t\right)\right|^2 dx\right)^{1/2} = \left(\varepsilon\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} |Y_t^2(z,t)|^2\, dz\right)^{1/2}.$$

Thanks to the explicit form (17) we have, for $0<\varepsilon\le\varepsilon_0$ small enough,

$$\begin{aligned}\max_{t\in[0,T]}\left(\varepsilon\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} |Y_t^2(z,t)|^2\, dz\right)^{1/2} &\le c\|y_{xxt}^0\|_{C([0,1]\times[0,T])}\left(\varepsilon\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} z^4\, dz\right)^{1/2} \\ &\le c\varepsilon^{-2}\varepsilon^{\frac{5\gamma}{2}}.\end{aligned}$$

It results that

$$\|I_\varepsilon^2\|_{C([0,T];L^2(0,1))} \le c\varepsilon^{\frac{5\gamma}{2}}. \tag{27}$$

Using Taylor expansions, for $1-x=\varepsilon z\to 0$, we have

$$\sum_{k=0}^2 \varepsilon^k y^k(x,t) = \sum_{k=0}^2 \varepsilon^k y^k(1-\varepsilon z,t) = \sum_{k=0}^2 \varepsilon^k\left(\sum_{i=0}^{2-k}\frac{1}{i!}\frac{\partial^i y^k}{\partial x^i}(1,t)(-\varepsilon z)^i\right) + \varepsilon^2\mathcal{O}\left((\varepsilon z)\right).$$

Since

$$Y^k(z,t) = Q^k(z,t) + e^{-Mz}P^k(z,t), \quad (z,t)\in(0,+\infty)\times(0,t),$$

with

$$P^k(z,t) = -\sum_{i=0}^k \frac{\partial^i y^k}{\partial x^i}(1,t)z^i, \qquad Q^k(z,t) = \sum_{i=0}^k (-1)^i\frac{\partial^i y^k}{\partial x^i}(1,t)z^i,$$

we have

$$\sum_{k=0}^2 \varepsilon^k Y^k(z,t) - \sum_{k=0}^2 \varepsilon^k y^k(1-\varepsilon z,t) = \varepsilon^2\mathcal{O}\left((\varepsilon z)\right) + e^{-Mz}\sum_{k=0}^2\varepsilon^k P^k(z,t). \tag{28}$$

Using the previous estimate we have

$$\begin{aligned}&\|I_\varepsilon^3\|_{C([0,T];L^2(0,1))} \\ &= M\varepsilon^{-\gamma}\left\|\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\left(\sum_{k=0}^2\varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},t\right) - \sum_{k=0}^2\varepsilon^k y^k(1-\varepsilon z,t)\right)\right\|_{C([0,T];L^2(0,1))} \\ &\le c\varepsilon^{2-\gamma}\left(\int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma}(1-x)^2\, dx\right)^{1/2} \\ &\le c\varepsilon^{2+\frac{\gamma}{2}}.\end{aligned} \tag{29}$$

Similarly we have

$$\|I_\varepsilon^4\|_{C([0,T];L^2(0,1))} \le c\varepsilon^{3-\frac{\gamma}{2}}. \tag{30}$$

It results from (28) that

$$\varepsilon^{-1}\sum_{k=0}^{2}\varepsilon^k Y_z^k(z,t)+\sum_{k=0}^{2}\varepsilon^k y_x^k(1-\varepsilon z,t)=\varepsilon\mathcal{O}\left((\varepsilon z)\right)+\varepsilon^{-1}e^{-Mz}\sum_{k=0}^{2}\varepsilon^k P_z^k(z,t)$$
$$-\varepsilon^{-1}Me^{-Mz}\sum_{k=0}^{2}\varepsilon^k P^k(z,t).$$

Arguing as for I_ε^3 we find that

$$\|I_\varepsilon^5\|_{C([0,T];L^2(0,1))}\leq c\varepsilon^{2+\frac{\gamma}{2}}. \tag{31}$$

Collecting estimates (26), (27), (29)–(31) we obtain (25). The proof of the lemma is complete. □

Let us now consider the initial layer corrector θ^ε defined as the solution of

$$\begin{cases}\theta_t^\varepsilon-\varepsilon\theta_{xx}^\varepsilon+M\theta_x^\varepsilon=0, & (x,t)\in Q_T,\\ \theta^\varepsilon(0,t)=\theta^\varepsilon(1,t)=0, & t\in(0,T),\\ \theta^\varepsilon(x,0)=\theta_0^\varepsilon(x), & x\in(0,1),\end{cases} \tag{32}$$

with

$$\theta_0^\varepsilon(x)=:y_0(x)-w_2^\varepsilon(x,0)=(1-\mathcal{X}_\varepsilon(x))\left(y_0(x)-\sum_{k=0}^{2}\varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},0\right)\right),\quad x\in(0,1). \tag{33}$$

The following lemma gives an estimate of $\|\theta^\varepsilon\|_{C([0,T];L^2(0,1))}$.

LEMMA 2.3 *Let θ^ε be the solution of problem (32), (33). Then there is a constant c independent of ε such that*

$$\|\theta^\varepsilon\|_{C([0,T];L^2(0,1))}\leq ce^{-\frac{M}{\varepsilon}(1-2\varepsilon^\gamma)}. \tag{34}$$

PROOF. Let $\alpha>0$. We define $\rho^\varepsilon(x,t)=e^{\frac{-M\alpha x}{2\varepsilon}}\theta^\varepsilon(x,t)$ then check that

$$L_\varepsilon\theta^\varepsilon=e^{\frac{M\alpha x}{2\varepsilon}}\left(\rho_t^\varepsilon-\varepsilon\rho_{xx}^\varepsilon+M(1-\alpha)\rho_x^\varepsilon-\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)\rho^\varepsilon\right)\quad\text{in } Q_T.$$

Consequently, ρ^ε is a solution of

$$\begin{cases}\rho_t^\varepsilon-\varepsilon\rho_{xx}^\varepsilon+M(1-\alpha)\rho_x^\varepsilon-\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)\rho^\varepsilon=0 & \text{in} & Q_T,\\ \rho^\varepsilon(0,\cdot)=\rho^\varepsilon(1,\cdot)=0 & \text{on} & (0,T),\\ \rho^\varepsilon(\cdot,0)=e^{\frac{-M\alpha x}{2\varepsilon}}\theta_0^\varepsilon & \text{in} & (0,1).\end{cases}$$

Multiplying the main equation by ρ^ε and integrating over $(0,1)$ then leads to

$$\frac{d}{dt}\|\rho^\varepsilon(\cdot,t)\|^2+2\varepsilon\|\rho_x^\varepsilon(\cdot,t)\|_{L^2(0,1)}^2=\frac{M^2}{2\varepsilon}(\alpha^2-2\alpha)\|\rho^\varepsilon(\cdot,t)\|_{L^2(0,1)}^2,$$

and then to the estimate $\|\rho^\varepsilon(\cdot,t)\|_{L^2(0,1)}\leq\|\rho^\varepsilon(\cdot,0)\|_{L^2(0,1)}e^{\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)t}$, equivalently, to

$$\|e^{-\frac{M\alpha x}{2\varepsilon}}\theta^\varepsilon(\cdot,t)\|_{L^2(0,1)}\leq\|e^{-\frac{M\alpha x}{2\varepsilon}}\theta^\varepsilon(\cdot,0)\|_{L^2(0,1)}e^{\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)t}.$$

Consequently,

$$\begin{aligned}\|\theta^\varepsilon(\cdot,t)\|_{L^2(0,1)}&=\|e^{\frac{M\alpha x}{2\varepsilon}}e^{-\frac{M\alpha x}{2\varepsilon}}\theta^\varepsilon(\cdot,t)\|_{L^2(0,1)}\leq\|e^{\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)}\|e^{-\frac{M\alpha x}{2\varepsilon}}\theta^\varepsilon(\cdot,t)\|_{L^2(0,1)}\\&\leq\|e^{\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)}\|e^{-\frac{M\alpha x}{2\varepsilon}}\theta^\varepsilon(\cdot,0)\|_{L^2(0,1)}e^{\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)t}\\&\leq\|e^{-\frac{M\alpha x}{2\varepsilon}}\theta^\varepsilon(\cdot,0)\|_{L^2(1-2\varepsilon^\gamma,1)}e^{\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)t}\\&\leq e^{\frac{-M\alpha(1-2\varepsilon^\gamma)}{2\varepsilon}}\|\theta_0^\varepsilon\|_{L^2(1-2\varepsilon^\gamma,1)}e^{\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)t}\\&\leq\|\theta_0^\varepsilon\|_{L^2(1-2\varepsilon^\gamma,1)}e^{-\frac{M\alpha}{2\varepsilon}\left(1-2\varepsilon^\gamma+\left(1-\frac{\alpha}{2}\right)Mt\right)},\end{aligned}$$

using that (recall that $\alpha > 0$) $\|e^{\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)} = e^{\frac{M\alpha}{2\varepsilon}}$ and $\|e^{-\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(1-2\varepsilon^\gamma,1)} = e^{\frac{-M\alpha(1-2\varepsilon^\gamma)}{2\varepsilon}}$. The value $\alpha = 2$ then leads to

$$\|\theta^\varepsilon\|_{C([0,T];L^2(0,1))} \leq \|\theta_0^\varepsilon\|_{L^2(1-2\varepsilon^\gamma,1)} e^{-\frac{M}{\varepsilon}(1-2\varepsilon^\gamma)}. \tag{35}$$

Let us now give an estimate of $\|\theta_0^\varepsilon\|_{L^2(1-2\varepsilon^\gamma,1)}$. Using (15)–(17) it holds that $\theta_0^\varepsilon = a^\varepsilon + b^\varepsilon$, with

$$a^\varepsilon(x) = (1-\mathcal{X}_\varepsilon(x))\left(y_0(x) - y_0(1) + y_0^{(1)}(1)(\varepsilon z) - y_0^{(2)}(1)\frac{(\varepsilon z)^2}{2}\right),$$
$$b^\varepsilon(x) = (1-\mathcal{X}_\varepsilon(x))\, e^{-Mz}\left(y_0(1) + y_0^{(1)}(1)(\varepsilon z) + y_0^{(2)}(1)\frac{(\varepsilon z)^2}{2}\right), \quad \left(z = \frac{1-x}{\varepsilon}\right).$$

Using Taylor's expansion, for $1 - x = \varepsilon z \to 0$, we have

$$a^\varepsilon(x) = (1-\mathcal{X}_\varepsilon(x))\,\frac{(1-x)^3}{6} y_0^{(3)}(\zeta), \quad \zeta \in (x,1),$$

hence $\|a^\varepsilon\|_{L^2(0,1)} \leq c\varepsilon^{\frac{7\gamma}{2}}$. Since

$$\|b^\varepsilon\|^2_{L^2(0,1)} \leq \int_{1-2\varepsilon^\gamma}^1 e^{-2Mz}\left(y_0(1) + y_0^{(1)}(1)(\varepsilon z) + y_0^{(2)}(1)\frac{(\varepsilon z)^2}{2}\right)^2 dx,$$

we have $\|b^\varepsilon\|_{L^2(0,1)} \leq c\varepsilon^{\frac{1}{2}}$. It results that

$$\|\theta_0^\varepsilon\|_{L^2(0,1)} \leq c\left(\varepsilon^{\frac{7\gamma}{2}} + \varepsilon^{\frac{1}{2}}\right), \tag{36}$$

then estimate (34) results from (35) and (36). □

Let us now establish the following result.

Lemma 2.4 *Let y^ε be the solution of problem* (7), *let w_2^ε be the function defined by* (21) *and thet θ^ε be the solution of problem* (32), (33). *Assume that the assumptions of Lemma 2.1 hold true. Then there is a constant c, independent of ε, such that*

$$\|y^\varepsilon - w_2^\varepsilon - \theta^\varepsilon\|_{C([0,T];L^2(0,1))} \leq c\varepsilon^{\frac{5\gamma}{2}}. \tag{37}$$

Proof. Let us consider the function $z^\varepsilon = y^\varepsilon - w_2^\varepsilon - \theta^\varepsilon$. It satisfies

$$\begin{cases} L_\varepsilon(z^\varepsilon) = -L_\varepsilon(w_2^\varepsilon), & (x,t) \in Q_T, \\ z^\varepsilon(0,t) = z^\varepsilon(1,t) = 0, & t \in (0,T), \\ z^\varepsilon(x,0) = 0, & x \in (0,1). \end{cases} \tag{38}$$

Multiplying equation (38) by z^ε, integrating by parts and using the Young inequality yields

$$\frac{1}{2}\|z^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} + \varepsilon\|z_x^\varepsilon\|^2_{L^2(0,1)\times 0,t))} = -\int_0^t\!\!\int_0^1 L_\varepsilon(w_2^\varepsilon) z^\varepsilon\, dxds$$
$$\leq \frac{1}{2}\int_0^t \|L_\varepsilon(w_2^\varepsilon)(\cdot,s)\|^2_{L^2(0,1)} ds + \frac{1}{2}\int_0^t \|z^\varepsilon(\cdot,s)\|^2_{L^2(0,1)} ds.$$

Gronwall's lemma then gives

$$\|z^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} \leq \|L_\varepsilon(w_2^\varepsilon)\|^2_{L^2(Q_T)} e^t, \quad \forall t \in (0,T], \tag{39}$$

then using Lemma 2.2 we get the estimate (37). □

Using Lemmas 2.3 and 2.4 we immediately obtain the following result.

Theorem 2.1 *Let y^ε be the solution of problem* (7) *and let w_2^ε be the function defined by* (21). *Assume that the assumptions of Lemma 2.1 hold true. Then there exist two positive constants c and ε_0, c independent of ε, such that, for any $0 < \varepsilon < \varepsilon_0$,*

$$\|y^\varepsilon - w_2^\varepsilon\|_{C([0,T];L^2(0,1))} \leq c\varepsilon^{\frac{5\gamma}{2}}. \tag{40}$$

2.3 High order asymptotic approximation

Here we construct an asymptotic approximation of the solution y^ε of (7) at any order m. The outer expansion is $\displaystyle\sum_{k=0}^{m} \varepsilon^k y^k(x,t)$, where the functions y^0 and y^k ($1 \le k \le m$) are given by (10) and (11), respectively. The inner expansion is given by $\displaystyle\sum_{k=0}^{m} \varepsilon^k Y^k(z,t)$, where the function Y^0 is given by (15), and the function Y^k ($1 \le k \le m$) is a solution of problem (18).

LEMMA 2.5 *For any $1 \le k \le m$, the solution of problem* (18) *reads*

$$Y^k(z,t) = Q^k(z,t) + e^{-Mz} P^k(z,t), \quad (z,t) \in (0,+\infty) \times (0,t), \tag{41}$$

where

$$P^k(z,t) = -\sum_{i=0}^{k} \frac{1}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1,t) z^i, \quad Q^k(z,t) = \sum_{i=0}^{k} \frac{(-1)^i}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1,t) z^i.$$

PROOF. We argue by induction on k. We have seen that (41) is valid for $k=1$. Then we assume the validity of the induction hypothesis for the integer k, and consider the function $Y^{k+1}(z,t)$ defined as

$$Y^{k+1}(z,t) = Q^{k+1}(z,t) + e^{-Mz} P^{k+1}(z,t), \quad (z,t) \in (0,+\infty) \times (0,t).$$

We have

$$Y_z^{k+1} = Q_z^{k+1} + e^{-Mz} P_z^{k+1} - M e^{-Mz} P^{k+1},$$

and

$$Y_{zz}^{k+1} = Q_{zz}^{k+1} + e^{-Mz} P_{zz}^{k+1} - 2M e^{-Mz} P_z^{k+1} + M^2 e^{-Mz} P^{k+1},$$

then

$$Y_{zz}^{k+1} + M Y_z^{k+1} = Q_{zz}^{k+1} + M Q_z^{k+1} + e^{-Mz} \left(P_{zz}^{k+1} - M P_z^{k+1} \right). \tag{42}$$

One can write $Q_{zz}^{k+1} + M Q_z^{k+1}$ in the form

$$\begin{aligned} Q_{zz}^{k+1} + M Q_z^{k+1} = & \sum_{i=2}^{k+1} \frac{(-1)^i}{(i-2)!} \left[\frac{\partial^i y^{k+1-i}}{\partial x^i}(1,t) - M \frac{\partial^{i-1} y^{k+2-i}}{\partial x^{i-1}}(1,t) \right] z^{i-2} \\ & + M \frac{(-1)^{k+1}}{k!} \frac{\partial^{k+1} y^0}{\partial x^{k+1}}(1,t) z^{k+1}. \end{aligned}$$

We deduce from equations (8) and (9) that

$$M \frac{\partial^{k+1} y^0}{\partial x^{k+1}} = -\frac{\partial^{k+1} y^0}{\partial x^k \partial t}, \quad \frac{\partial^i y^{k+1-i}}{\partial x^i} - M \frac{\partial^{i-1} y^{k+2-i}}{\partial x^{i-1}} = \frac{\partial^{i-1} y^{k+2-i}}{\partial x^{i-2} \partial t}.$$

It results that

$$\begin{aligned} Q_{zz}^{k+1} + M Q_z^{k+1} &= \sum_{i=2}^{k+1} \frac{(-1)^i}{(i-2)!} \frac{\partial^{i-1} y^{k+2-i}}{\partial x^{i-2} \partial t}(1,t)\, z^{i-2} + \frac{(-1)^k}{k!} \frac{\partial^{k+1} y^0}{\partial x^k \partial t}(1,t) z^k \\ &= \sum_{i=2}^{k+2} \frac{(-1)^i}{(i-2)!} \frac{\partial^{i-1} y^{k+2-i}}{\partial x^{i-2} \partial t}(1,t)\, z^{i-2} \\ &= \frac{\partial}{\partial t} \sum_{i=0}^{k} \frac{(-1)^i}{i!} \frac{\partial^i y^{k-i}}{\partial x^i}(1,t) z^i = Q_t^k(z,t). \end{aligned} \tag{43}$$

Similar calculations allow to prove that

$$P_{zz}^{k+1} - M P_z^{k+1} = P_t^k. \tag{44}$$

From (42)–(44) we deduce that $Y_{zz}^{k+1} + MY_z^{k+1} = Y_t^k$. We conclude that the function Y^k defined by (41) satisfies equation $(18)_1$ for any $1 \le k \le m$. Moreover, Y^k satisfies the conditions $(18)_2$ and $(18)_3$. That completes the proof of the lemma. □

We then introduce the function

$$w_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^{m} \varepsilon^k y^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{m} \varepsilon^k Y^k \left(\frac{1-x}{\varepsilon}, t \right), \tag{45}$$

defined to be an asymptotic approximation at order m of the solution y^ε of (7). Function $\mathcal{X}_\varepsilon$ is defined on (20). To justify all the computations we will perform we need some regularity assumptions on the data y_0, v^0, v^1, $\cdots$, v^m. We have the following result.

Lemma 2.6 *(i) Assume that $y_0 \in C^{2m+1}[0,1]$, $v^0 \in C^{2m+1}[0,T]$ and the following C^{2m+1}-matching conditions are satisfied*

$$M^p (y_0)^{(p)}(0) + (-1)^{p+1} (v^0)^{(p)}(0) = 0, \quad 0 \le p \le 2m+1. \tag{46}$$

Then the function y^0 defined by (10) belongs to $C^{2m+1}(\overline{Q_T})$.
(ii) Additionally, assume that $v^k \in C^{2(m-k)+1}[0,T]$, and the following $C^{2(m-k)+1}$-matching conditions are satisfied, respectively,

$$(v^k)^{(p)}(0) = \sum_{i+j=p-1} (-1)^i M^i \frac{\partial^{p+1} y^{k-1}}{\partial x^{i+2} \partial t^j}(0,0), \quad 0 \le p \le 2(m-k)+1. \tag{47}$$

Then the function y^k belongs to $C^{2(m-k)+1}(\overline{Q_T})$.

Proof. *(i)* For the proof of (46) we refer to that of (22).
(ii) Using a change of variable we rewrite (11) in the form

$$y^k(x,t) = \begin{cases} \displaystyle\int_0^t y_{xx}^{k-1}\left(x + M(s-t), s\right) ds, & x > Mt, \\ \displaystyle v^k \left(t - \frac{x}{M} \right) + \int_{t-x/M}^t y_{xx}^{k-1}\left(x + M(s-t), s\right) ds, & x < Mt. \end{cases} \tag{48}$$

For notational convenience we omit in the sequel the index k and denote $y_{xx}^{k-1} = f$ so that (48) reads

$$y(x,t) = \begin{cases} \displaystyle\int_0^t f\left(x + (s-t)M, s\right) ds, & x > Mt, \\ \displaystyle v(t - \frac{x}{M}) + \int_{t-x/M}^t f\left(x + M(s-t), s\right) ds, & x < Mt. \end{cases} \tag{49}$$

Differentiating (49) with respect to x we have

$$\frac{\partial y}{\partial x}(x,t) = \begin{cases} \displaystyle\int_0^t \frac{\partial f}{\partial x}\left(x + (s-t)M, s\right) ds, & x > Mt, \\ \displaystyle -\frac{1}{M} v'(t - \frac{x}{M}) + \int_{t-x/M}^t \frac{\partial f}{\partial x}\left(x + M(s-t), s\right) ds + \frac{1}{M} f\left(0, t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

Differentiating once again we have

$$\frac{\partial^2 y}{\partial x^2}(x,t) = \begin{cases} \displaystyle\int_0^t \frac{\partial^2 f}{\partial x^2}\left(x + (s-t)M, s\right) ds, & x > Mt, \\ \displaystyle \frac{1}{M^2} v''(t - \frac{x}{M}) + \int_{t-x/M}^t \frac{\partial^2 f}{\partial x^2}\left(x + M(s-t), s\right) ds + \frac{1}{M} f_x\left(0, t - \frac{x}{M}\right) & \\ \displaystyle -\frac{1}{M^2} f_t\left(0, t - \frac{x}{M}\right), & x < Mt. \end{cases}$$

Successive partial derivatives with respect to x lead to the formulae:

$$\frac{\partial^p y}{\partial x^p}(x,t) = \int_0^t \frac{\partial^p f}{\partial x^p}\left(x + M(s-t), s\right) ds \qquad \text{for } x > Mt, \tag{50}$$

and

$$\begin{aligned}\frac{\partial^p y}{\partial x^p}(x,t) =& \frac{(-1)^p}{M^p} v^{(p)}\left(t - \frac{x}{M}\right) + \int_{t-x/M}^t \frac{\partial^p f}{\partial x^p}\left(x + M(s-t), s\right) ds \\ &+ \sum_{i+j=p-1} \frac{(-1)^j}{M^{j+1}} \frac{\partial^{p-1} f}{\partial x^i \partial t^j}\left(0, t - \frac{x}{M}\right) \qquad \text{for } x < Mt.\end{aligned} \tag{51}$$

These formulae can be easily justified by induction. Then it results from (50) and (51) that $\frac{\partial^p y}{\partial x^p}$ is continuous in $\overline{Q_T}$ if

$$\frac{(-1)^p}{M^p} v^{(p)}(0) = - \sum_{i+j=p-1} \frac{(-1)^j}{M^{j+1}} \frac{\partial^{p-1} f}{\partial x^i \partial t^j}(0,0)\,,$$

which is equivalent to (47). Similar calculations allow to establish the formulae

$$\begin{aligned}\frac{\partial^p y}{\partial t^p}(x,t) =& (-1)^p M^p \int_0^t \frac{\partial^p f}{\partial x^p}\left(x + M(s-t), s\right) ds \\ &+ \sum_{i+j=p-1} (-1)^i M^i \frac{\partial^{p-1} f}{\partial x^i \partial t^j}(x,t) \qquad \text{for } x > Mt,\end{aligned} \tag{52}$$

and

$$\begin{aligned}\frac{\partial^p y}{\partial t^p}(x,t) =& v^{(p)}\left(t - \frac{x}{M}\right) + (-1)^p M^p \int_{t-x/M}^t \frac{\partial^p f}{\partial x^p}\left(x + M(s-t), s\right) ds \\ &+ \sum_{i+j=p-1} (-1)^i M^i \left(\frac{\partial^{p-1} f}{\partial x^i \partial t^j}(x,t) - \frac{\partial^{p-1} f}{\partial x^i \partial t^j}\left(t - \frac{x}{M}\right)\right) \qquad \text{for } x < Mt.\end{aligned} \tag{53}$$

It results from (52) and (53) that $\frac{\partial^p y}{\partial t^p}$ is continuous in $\overline{Q_T}$ if

$$v^{(p)}(0) = \sum_{i+j=p-1} (-1)^i M^i \frac{\partial^{p-1} f}{\partial x^i \partial t^j}(0,0)\,,$$

that is the condition (47). Using equation (9) we easily verify that the mixed partial derivatives, of order $0 \le p \le 2(m-k)+1$, of y^k are continuous under condition (47). $\square$

Remark 2 *For $m = 2$ and $k = 1$ the conditions (47) read*

$$\begin{aligned}&v^1(0) = 0, \quad (v^1)^{(1)}(0) = y^0_{xx}(0,0) = y_0^{(2)}(0) = M^{-2}(v^0)^{(2)}(0),\\ &(v^1)^{(2)}(0) = y^0_{xxt}(0,0) - M y^0_{xxx} = -2M y_0^{(3)}(0) = 2M^{-2}(v^0)^{(3)}(0),\\ &(v^1)^{(3)}(0) = M^2 y^0_{xxxx} - M y^0_{xxxt}(0,0) + y^0_{xxtt}(0,0) = 3M^2 y^{(4)}(0) = 3M^{-2} v^{(4)}(0).\end{aligned}$$

For $k = 2$ we have

$$v^2(0) = 0, \quad (v^2)^{(1)}(0) = y^1_{xx}(0,0) = 0.$$

Thus we retrieve the matching conditions (23) and (24).

Let us now establish the following result.

Lemma 2.7 *Let w^ε_m be the function defined by (45). Assume that the assumptions of Lemma 2.6 hold true. Then there is a constant c_m independent of ε such that*

$$\|L_\varepsilon(w^\varepsilon_m)\|_{C([0,T];L^2(0,1))} \le c_m \varepsilon^{\frac{(2m+1)\gamma}{2}}\,. \tag{54}$$

Proof. A straightforward calculation gives

$$L_\varepsilon(w_m^\varepsilon)(x,t) = \sum_{i=1}^{5} J_\varepsilon^i(x,t), \tag{55}$$

with

$$\begin{aligned}
J_\varepsilon^1(x,t) &= -\varepsilon^{m+1} y_{xx}^m(x,t)\mathcal{X}_\varepsilon(x),\\
J_\varepsilon^2(x,t) &= \varepsilon^m(1-\mathcal{X}_\varepsilon(x))Y_t^m\left(\frac{1-x}{\varepsilon},t\right),\\
J_\varepsilon^3(x,t) &= M\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{-\gamma}\left(\sum_{k=0}^m \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},t\right) - \sum_{k=0}^m \varepsilon^k y^k(x,t)\right),\\
J_\varepsilon^4(x,t) &= \mathcal{X}''\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-2\gamma}\left(\sum_{k=0}^m \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},t\right) - \sum_{k=0}^m \varepsilon^k y^k(x,t)\right),\\
J_\varepsilon^5(x,t) &= 2\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-\gamma}\left(\varepsilon^{-1}\sum_{k=0}^m \varepsilon^k Y_z^k\left(\frac{1-x}{\varepsilon},t\right) + \sum_{k=0}^m \varepsilon^k y_x^k(x,t)\right).
\end{aligned}$$

Clearly,

$$\|J_\varepsilon^1\|_{L^\infty(0,T;L^2(0,1))} \le \varepsilon^{m+1}\|y_{xx}^m\|_{C([0,T];L^2(0,1))} \le c_m\varepsilon^{m+1}, \tag{56}$$

and

$$\begin{aligned}
\|J_\varepsilon^2\|_{C([0,T];L^2(0,1))} &\le \varepsilon^m \left\|(1-\mathcal{X}_\varepsilon(x))Y_t^m\left(\frac{1-x}{\varepsilon},t\right)\right\|_{C([0,T];L^2(0,1))}\\
&\le \varepsilon^m \max_{t\in[0,T]}\left(\int_{1-2\varepsilon^\gamma}^1 \left|Y_t^m\left(\frac{1-x}{\varepsilon},t\right)\right|^2 dx\right)^{1/2}\\
&\le \varepsilon^m \max_{t\in[0,T]}\left(\varepsilon\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} |Y_t^m(z)|^2\, dz\right)^{1/2}.
\end{aligned}$$

Thanks to the explicit form (41) we have, for $0<\varepsilon\le\varepsilon_0$ small enough,

$$\begin{aligned}
\max_{t\in[0,T]}\left(\varepsilon\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} |Y_t^m(z,t)|^2\, dz\right)^{1/2} &\le c_m\left\|\frac{\partial^{m+1}y^0}{\partial x^m\partial t}\right\|_{C([0,1]\times[0,T])}\left(\varepsilon\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} z^{2m}\, dz\right)^{1/2}\\
&\le c_m\varepsilon^{-m}\varepsilon^{\frac{2m+1}{2}\gamma}.
\end{aligned}$$

It results that

$$\|J_\varepsilon^2\|_{C([0,T];L^2(0,1))} \le c_m\varepsilon^{\frac{2m+1}{2}\gamma}. \tag{57}$$

Using Taylor expansions, for $1-x=\varepsilon z\to 0$, we have

$$\sum_{k=0}^m \varepsilon^k y^k(x,t) = \sum_{k=0}^m \varepsilon^k y^k(1-\varepsilon z,t) = \sum_{k=0}^m \varepsilon^k\left(\sum_{i=0}^{m-k}\frac{1}{i!}\frac{\partial^i y^k}{\partial x^i}(1,t)(-\varepsilon z)^i\right) + \varepsilon^m\mathcal{O}\left((\varepsilon z)\right).$$

According to (41) it results that

$$\sum_{k=0}^m \varepsilon^k Y^k(z,t) - \sum_{k=0}^m \varepsilon^k y^k(1-\varepsilon z,t) = \varepsilon^m\mathcal{O}\left((\varepsilon z)\right) + e^{-Mz}\sum_{k=0}^m \varepsilon^k P^k(z,t). \tag{58}$$

Using the previous estimate we have

$$\begin{aligned}\|J_\varepsilon^3\|_{C([0,T];L^2(0,1))} &= M\varepsilon^{-\gamma}\left\|\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\left(\sum_{k=0}^m \varepsilon^k Y^k(z,t)-\sum_{k=0}^m \varepsilon^k y^k(1-\varepsilon z,t)\right)\right\|_{C([0,T];L^2(0,1))}\\ &\le c_m\varepsilon^{m-\gamma}\left(\int_{1-2\varepsilon^\gamma}^{1-\varepsilon^\gamma}(1-x)^2\,dx\right)^{1/2}\\ &\le c_m\varepsilon^{m+\frac{\gamma}{2}}. \end{aligned}\tag{59}$$

Similarly we have

$$\|J_\varepsilon^4\|_{C([0,T];L^2(0,1))}\le c_m\varepsilon^{(m+1)-\frac{\gamma}{2}}.\tag{60}$$

It results from (58) that

$$\begin{aligned}\varepsilon^{-1}\sum_{k=0}^m\varepsilon^k Y_z^k(z,t)+\sum_{k=0}^m\varepsilon^k y_x^k(1-\varepsilon z,t)=&\varepsilon^{m-1}\mathcal{O}\left((\varepsilon z)\right)+\varepsilon^{-1}e^{-Mz}\sum_{k=0}^m\varepsilon^k P_z^k(z,t)\\ &-\varepsilon^{-1}Me^{-Mz}\sum_{k=0}^m\varepsilon^k P^k(z,t).\end{aligned}$$

Arguing as for J_ε^3 we deduce that

$$\|J_\varepsilon^5\|_{C([0,T];L^2(0,1))}\le c_m\varepsilon^{m+\frac{\gamma}{2}}.\tag{61}$$

Collecting estimates (56), (57), (59)–(61) we obtain (54). The proof of the lemma is complete. □

We define the initial layer corrector θ_m^ε as the solution of

$$\begin{cases}\theta_{mt}^\varepsilon-\varepsilon\theta_{mxx}^\varepsilon+M\theta_{mx}^\varepsilon=0, & (x,t)\in(0,1)\times(0,T),\\ \theta_m^\varepsilon(0,t)=\theta_m^\varepsilon(1,t)=0, & t\in(0,T),\\ \theta_m^\varepsilon(x,0)=\theta_{m0}^\varepsilon(x), & x\in(0,1),\end{cases}\tag{62}$$

with

$$\theta_{m0}^\varepsilon(x)=:y_0(x)-w_m^\varepsilon(x,0)=(1-\mathcal{X}_\varepsilon(x))\left(y_0(x)-\sum_{k=0}^m\varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},0\right)\right),\quad x\in(0,1).\tag{63}$$

We have the analog of Lemma 2.3.

Lemma 2.8 *Let θ_m^ε be the solution of problem (62), (63). Then there exist a constant c_m, independent of ε, such that*

$$\|\theta_m^\varepsilon\|_{C([0,T];L^2(0,1))}\le c_m e^{-\frac{M}{\varepsilon}(1-2\varepsilon^\gamma)}.\tag{64}$$

Proof. The proof follows that of Lemma 2.3. We have (see (35))

$$\|\theta_m^\varepsilon\|_{C([0,T];L^2(0,1))}\le\|\theta_{m0}^\varepsilon\|_{L^2(1-2\varepsilon^\gamma,1)}e^{-\frac{M}{\varepsilon}(1-2\varepsilon^\gamma)}.\tag{65}$$

Let us now give an estimate of $\|\theta_{m0}^\varepsilon\|_{L^2(1-2\varepsilon^\gamma,1)}$. Using Lemma 2.5 it holds that $\theta_{m0}^\varepsilon=a_m^\varepsilon+b_m^\varepsilon$, with

$$\begin{aligned}a_m^\varepsilon(x)&=(1-\mathcal{X}_\varepsilon(x))\left(y_0(x)-\sum_{i=0}^m\frac{(-1)^i}{i!}y_0^{(i)}(1)(\varepsilon z)^i\right),\\ b_m^\varepsilon(x)&=(1-\mathcal{X}_\varepsilon(x))\,e^{-Mz}\sum_{i=0}^m\frac{1}{i!}y_0^{(i)}(1)(\varepsilon z)^i,\quad\left(z=\frac{1-x}{\varepsilon}\right).\end{aligned}$$

Using Taylor's expansion, for $1-x=\varepsilon z \to 0$, we have

$$a_m^\varepsilon(x) = (1-\mathcal{X}_\varepsilon(x))\,\frac{(1-x)^{m+1}}{(m+1)!}\,y_0^{(m+1)}(\zeta_1), \quad \zeta_1 \in (x,1),$$

hence $\|a_m^\varepsilon\|_{L^2(0,1)} \le c_m \varepsilon^{\frac{(2m+3)\gamma}{2}}$. We also have $\|b^\varepsilon\|_{L^2(0,1)} \le c_m \varepsilon^{\frac{1}{2}}$. It results that

$$\|\theta_0^\varepsilon\|_{L^2(0,1)} \le c_m \left(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{(2m+3)\gamma}{2}}\right), \tag{66}$$

then estimate (64) results from (65) and (66). □

Arguing as in Section 2.2 one can establish the analog of Lemma 2.4.

LEMMA 2.9 *Let y^ε be the solution of problem* (7), *let w_m^ε be the function defined by* (45) *and thet θ_m^ε be the solution of problem* (62), (63). *Assume that the assumptions of Lemma 2.1 hold true. Then there is a constant c_m, independent of ε, such that*

$$\|y^\varepsilon - w_m^\varepsilon - \theta_m^\varepsilon\|_{C([0,T];L^2(0,1))} \le c_m \varepsilon^{\frac{2m+1}{2}\gamma}.$$

Using Lemmas 2.8 and 2.9 we readily obtain the following result.

THEOREM 2.2 *Let y^ε be the solution of problem* (7) *and let w_m^ε be the function defined by* (45). *Assume that the assumptions of Lemma 2.6 hold true. Then there exist two positive constants c and ε_0, c independent of ε, such that, for any $0<\varepsilon<\varepsilon_0$,*

$$\|y^\varepsilon - w_m^\varepsilon\|_{C([0,T];L^2(0,1))} \le c_m \varepsilon^{\frac{2m+1}{2}\gamma}. \tag{67}$$

We have thus constructed a regular and strongly convergent approximation (as $\varepsilon \to 0$) w_m^ε of y^ε, unique solution of (7).

2.4 Passing to the limit as $m \to \infty$. Particular case

Our objective here is to show that, under some strong conditions on the initial condition y_0 and the functions v^k, we can pass to the limit with respect to the parameter m and establish a convergence result of the sequence $(w_m^\varepsilon)_m$. We make the following assumptions:

(i) The initial condition y_0 belongs to $C^\infty[0,1]$ and there are $c_0, b \in \mathbb{R}$ such that

$$\|y_0^{(m)}\|_{C[0,1]} \le c_0 b^m, \quad \forall m \in \mathbb{N}. \tag{68}$$

(ii) $(v^k)_{k\ge 0}$ is a sequence of polynomials of degree $\le p-1$, $p \ge 1$, uniformly bounded in $C^{p-1}[0,T]$.

(iii) For any $k \in \mathbb{N}$, for any $m \in \mathbb{N}$, the functions v^k and y_0 satisfy the matching conditions of Lemma 2.6.

We establish the following result.

THEOREM 2.3 *Let, for any $m \in \mathbb{N}$, y_m^ε denote the solution of problem* (7), *and w_m^ε the function defined by* (45). *We assume that the assumptions (i)–(iii) hold true. Then, there exist $\varepsilon_0 > 0$ and a function $\tilde\theta^\varepsilon \in C^\infty(\overline{Q_T})$ satisfying an exponential decay, such that, for any fixed $0<\varepsilon<\varepsilon_0$, we have*

$$y_m^\varepsilon - w_m^\varepsilon - \tilde\theta^\varepsilon \to 0 \quad \text{in } C([0,T],L^2(0,1)), \quad \text{as } m \to +\infty.$$

Consequently

$$\begin{aligned}\lim_{m\to+\infty} w_m^\varepsilon(x,t) &= \mathcal{X}_\varepsilon(x)\sum_{k=0}^\infty \varepsilon^k y^k(x,t) + (1-\mathcal{X}_\varepsilon(x))\sum_{k=0}^\infty \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, t\right) + \tilde\theta^\varepsilon(x)\\ &= y^\varepsilon(x,t) + \tilde\theta^\varepsilon(x) \quad \textit{a.e. in } Q_T,\end{aligned}$$

where y^ε is the solution of problem (7) with $(7)_2$ replaced by $y^\varepsilon(0,t) = \sum_{k=0}^{\infty} \varepsilon^k v^k(t), \quad t \in (0,T)$. The function $\tilde{\theta}^\varepsilon$ satisfies

$$\|\tilde{\theta}^\varepsilon\|_{C([0,T],L^2(0,1))} \leq ce^{-M\frac{\varepsilon^\gamma}{\varepsilon}},$$

where c is a constant independent of ε.

Before proving this convergence with respect to the order m, we introduce the following lemma.

Lemma 2.10 *For $x < Mt$, the function y^m given by (10), (11) may be written in the form*

$$y^m(x,t) = v^m\left(t - \frac{x}{M}\right) + \sum_{j=1}^{m}\sum_{i=1}^{j} X_j^i(x)(v^{m-j})^{(i+j)}\left(t - \frac{x}{M}\right), \tag{69}$$

where, for any $1 \leq i \leq j \leq m$, $X_j^i(x)$ is a polynomial of degree $\leq i$.

Proof. Formula (69) is valid for $m = 2$ (see (13)). Assume the validity of the induction hypothesis for the integer m. Differentiating (69) twice with respect to x and using the equality

$$y^{m+1}(x,t) = v^{m+1}\left(t - \frac{x}{M}\right) + \int_0^{x/M} y_{xx}^m(sM, t - \frac{x}{M} + s)ds,$$

we get

$$\begin{aligned} y^{m+1}(x,t) =& v^{m+1}\left(t - \frac{x}{M}\right) + \frac{x}{M^3}(v^m)^{(2)}\left(t - \frac{x}{M}\right) + \sum_{j=1}^{m}\sum_{i=1}^{j} A_{j+1}^{i-1}(x)(v^{m-j})^{(i+j)}\left(t - \frac{x}{M}\right) \\ &+ \sum_{j=1}^{m}\sum_{i=1}^{j} B_{j+1}^{i}(x)(v^{m-j})^{(i+j+1)}\left(t - \frac{x}{M}\right) + \sum_{j=1}^{m}\sum_{i=1}^{j} C_{j+1}^{i+1}(x)(v^{m-j})^{(i+j+2)}\left(t - \frac{x}{M}\right), \end{aligned}$$

with

$$A_{j+1}^{i-1}(x) = \int_0^{\frac{x}{M}} (X_j^i)^{(2)}(s)\,ds, \quad B_{j+1}^{i}(x) = \int_0^{\frac{x}{M}} \frac{-2}{M}(X_j^i)^{(1)}(s)\,ds, \quad C_{j+1}^{i+1}(x) = \int_0^{\frac{x}{M}} \frac{1}{M^2}X_j^i(s)\,ds.$$

Clearly, $A_{j+1}^{i-1}(x)$ is a polynomial of degree $\leq i-1$, $B_{j+1}^{i}(x)$ is a polynomial of degree $\leq i$, and $C_{j+1}^{i+1}(x)$ is a polynomial of degree $\leq i+1$. Changing index of summation we can write

$$\begin{aligned} &\sum_{j=1}^{m}\sum_{i=1}^{j} A_{j+1}^{i-1}(x)(v^{m-j})^{(i+j)}\left(t - \frac{x}{M}\right) = \sum_{j=2}^{m+1}\sum_{i=1}^{j-2} A_{j}^{i}(x)(v^{m+1-j})^{(i+j)}\left(t - \frac{x}{M}\right), \\ &\sum_{j=1}^{m}\sum_{i=1}^{j} B_{j+1}^{i}(x)(v^{m-j})^{(i+j+1)}\left(t - \frac{x}{M}\right) = \sum_{j=2}^{m+1}\sum_{i=1}^{j-1} B_{j}^{i}(x)(v^{m+1-j})^{(i+j)}\left(t - \frac{x}{M}\right), \\ &\sum_{j=1}^{m}\sum_{i=1}^{j} C_{j+1}^{i+1}(x)(v^{m-j})^{(i+j+2)}\left(t - \frac{x}{M}\right) = \sum_{j=2}^{m+1}\sum_{i=2}^{j} C_{j}^{i}(x)(v^{m+1-j})^{(i+j)}\left(t - \frac{x}{M}\right). \end{aligned}$$

Let us set

$$\begin{aligned} &A_j^j(x) = 0 \ \text{ for } j = 1, \cdots, m+1, \quad A_{j+1}^j(x) = 0 \ \text{ for } j = 1, \cdots, m, \\ &B_j^j(x) = 0 \ \text{ for } j = 1, \cdots, m+1, \\ &C_1^1(x) = \frac{x}{M^3}, \qquad C_j^1(x) = 0 \ \text{ for } j = 2, \cdots, m+1. \end{aligned}$$

Then we can write

$$y^{m+1}(x,t) = v^{m+1}\left(t - \frac{x}{M}\right) + \sum_{j=1}^{m+1}\sum_{i=1}^{j} \tilde{X}_j^i(x)(v^{m+1-j})^{(i+j)}\left(t - \frac{x}{M}\right),$$

where

$$\tilde{X}_j^i(x) = A_j^i(x) + B_j^i(x) + C_j^i(x).$$

That completes the proof of formula (69) by induction. □

Proof (of Theorem 2.3-) Recall that (see (55)) $L_\varepsilon(w_m^\varepsilon)(x,t) = \sum_{i=1}^5 J_\varepsilon^i(x,t)$. We define

$$f_m^\varepsilon = f_{m,1}^\varepsilon + f_{m,2}^\varepsilon + f_{m,3}^\varepsilon \quad \text{in } Q_T,$$

with

$$f_{m,1}^\varepsilon(x,t) = -M\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{-\gamma}e^{-M\frac{1-x}{\varepsilon}}\sum_{k=0}^{m}\varepsilon^k P^k\left(\frac{1-x}{\varepsilon}, t\right),$$

$$f_{m,2}(x,t = -\mathcal{X}''\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-2\gamma}e^{-M\frac{1-x}{\varepsilon}}\sum_{k=0}^{m}\varepsilon^k P^k\left(\frac{1-x}{\varepsilon}, t\right),$$

$$f_{m,3}(x,t) = -2\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{-\gamma}e^{-M\frac{1-x}{\varepsilon}}\left(\sum_{k=0}^{m}\varepsilon^k P_z^k\left(\frac{1-x}{\varepsilon}, t\right) - M\sum_{k=0}^{m}\varepsilon^k P^k(\frac{1-x}{\varepsilon}, t)\right).$$

We also define

$$\tilde{\theta}_{m0}^\varepsilon(x) = (1 - \mathcal{X}_\varepsilon(x))\left(y_0(x) - \sum_{k=0}^{m}\varepsilon^k Y^k\left(\frac{1-x}{\varepsilon}, 0\right)\right), \quad x \in (0,1).$$

Let $\tilde{\theta}_m^\varepsilon$ be the solution of problem

$$\begin{cases} L_\varepsilon(\tilde{\theta}_m^\varepsilon) = f_m^\varepsilon, & (x,t) \in Q_T), \\ \tilde{\theta}_m^\varepsilon(0,t) = \tilde{\theta}_m^\varepsilon(1,t) = 0, & t \in (0,T), \\ \tilde{\theta}_m^\varepsilon(x,0) = \tilde{\theta}_{m0}^\varepsilon(x), & x \in (0,1). \end{cases} \tag{70}$$

Then the function $z_m^\varepsilon =: y_m^\varepsilon - w_m^\varepsilon - \tilde{\theta}_m^\varepsilon$ satisfies

$$\begin{cases} L_\varepsilon(z_m^\varepsilon) = -L_\varepsilon(w_m^\varepsilon) - f_m^\varepsilon, & \text{in } Q_T, \\ z_m^\varepsilon(0,t) = z_m^\varepsilon(1,t) = 0, & t \in (0,T), \\ z_m^\varepsilon(x,0) = 0, & x \in (0,1). \end{cases}$$

Multiplying the previous equation by z_m^ε and integrating by parts we get that

$$\int_0^1 |z_m^\varepsilon(x,t)|^2\,dx \le d_m^\varepsilon e^t, \qquad d_m^\varepsilon =: \int_0^T\int_0^1 \left|L_\varepsilon(w_m^\varepsilon)(x,s) + f_m^\varepsilon(x,s)\right|^2 dxds. \tag{71}$$

Let us verify that $(d_m^\varepsilon)_{m>0}$ tends to 0, as $m \to \infty$. We note that

$$L_\varepsilon(w_m^\varepsilon) + f_m^\varepsilon = J_\varepsilon^1 + J_\varepsilon^2 + (J_\varepsilon^3 + f_{m,1}) + (J_\varepsilon^4 + f_{m,2}) + (J_\varepsilon^5 + f_{m,3}).$$

- Estimate of $\|J_\varepsilon^1\|_{C([0,T],L^2(0,1))}$ - It is easily seen that

$$y_{xx}^m(x,t) = \frac{t^m}{m!}\,y_0^{(2m+2)}(x - Mt) \quad \text{for } x > Mt. \tag{72}$$

Using (68) we have

$$\max_{x\geq Mt} |y^m_{xx}(x,t)| \leq c_0 b^2 \frac{(b^2T)^m}{m!}.$$

We deduce that there is a constant c_1, independent of m, such that

$$\max_{x\geq Mt} |J^1_\varepsilon(x,t)| \leq c_1 \varepsilon^{m+1}. \tag{73}$$

For $x \leq Mt$, it results from (69) that y^m is a polynomial of degree $\leq p-1$ and, for large m ($m > p$),

$$y^m(x,t) = v^m\left(t - \frac{x}{M}\right) + \sum_{j=1}^{p-1}\sum_{i=1}^{j} X^i_j(x)(v^{m-j})^{(i+j)}\left(t - \frac{x}{M}\right).$$

Since all the terms in the right-hand side of the previous inequality are uniformly bounded in the space $C^{p-1}\left(\{(x,t) \in \overline{Q_T} : x \leq Mt\}\right)$, we deduce that there is a constant $c_2(p)$ independent of m such that

$$\max_{x\leq Mt} |y^m_{xx}(x,t)| \leq c_2(p),$$

then

$$\max_{x\leq Mt} |J^1_\varepsilon(x,t)| \leq c_2(p)\varepsilon^{m+1}. \tag{74}$$

It results from (73) and (74) that

$$\max_{(x,t)\overline{Q_T}} |J^1_\varepsilon(x,t)| \leq c_3(p)\varepsilon^{m+1}, \qquad c_3(p) = \max(c_1, c_2(p)). \tag{75}$$

• Estimate of $\|J^2_\varepsilon\|_{C([0,T],L^2(0,1))}$ - We have (see the proof of Lemma 2.7)

$$\|J^2_\varepsilon\|_{C([0,T];L^2(0,1))} \leq \varepsilon^m \max_{t\in[0,T]} \left(\varepsilon \int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} |Y^m_t(z,t)|^2\, dz\right)^{1/2}.$$

Thanks to Lemma 2.5 we have

$$Y^m(z,t) = Q^m(z,t) + e^{-Mz}P^m(z,t), \quad (z,t) \in (0,+\infty)\times(0,t),$$

where

$$P^m(z,t) = -\sum_{i=0}^{m} \frac{1}{i!}\frac{\partial^i y^{m-i}}{\partial x^i}(1,t)z^i, \quad Q^m(z,t) = \sum_{i=0}^{m} \frac{(-1)^i}{i!}\frac{\partial^i y^{m-i}}{\partial x^i}(1,t)z^i.$$

We have, for $x \geq Mt$,

$$\frac{\partial^{i+1} y^{m-i}}{\partial x^i \partial t}(x,t) = -M\frac{t^{m-i}}{(m-i)!}\, y_0^{(2m-i+1)}(x-Mt) + \frac{t^{m-i-1}}{(m-i-1)!}\, y_0^{(2m-i)}(x-Mt).$$

We deduce by using (68) that

$$\max_{x\geq Mt}\left|\frac{\partial^{i+1} y^{m-i}}{\partial x^i \partial t}(x,t)\right| \leq c_0(MTb+1)b^{m+1}\frac{(Tb)^{m-i-1}}{(m-i-1)!} \quad \text{ for } x \geq Mt. \tag{76}$$

For $x \leq Mt$, writing $y^{m-i}(x,t)$ in the form (69), we deduce that there is a constant $c_4(p)$ independent of m such that

$$\max_{x\leq Mt}\left|\frac{\partial^{i+1} y^{m-i}}{\partial x^i \partial t}(x,t)\right| \leq c_4(p). \tag{77}$$

We easily verify that there is $\varepsilon_1 > 0$ such that, for $0 < \varepsilon < \varepsilon_1$, we have

$$\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} z^{2i}\,dz \leq \int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} z^{2m}\,dz = \frac{2^{2m+1}}{2m+1}\frac{\varepsilon^{(2m+1)\gamma}}{\varepsilon^{2m+1}}. \tag{78}$$

Using (76)–(78) we deduce the estimate

$$\begin{aligned}
\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} |Y_t^m(z,t)|^2\,dz &\leq 2\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} \left(|Q_t^m(z,t)|^2 + |P_t^m(z,t)|^2\right)\,dz \\
&\leq c_5(p)\left(mb^{2m+2}\sum_{i=0}^{m-1}\left(\frac{(Tb)^{m-i-1}}{(m-i-1)!}\right)^2 + 1\right)\left(\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} z^{2m}\,dz\right) \\
&\leq c_6(p)\left(mb^{2m+2}+1\right)\left(\frac{2^{2m+1}}{2m+1}\frac{\varepsilon^{(2m+1)\gamma}}{\varepsilon^{2m+1}}\right) \\
&\leq \frac{c_6(p)}{\varepsilon^{2m+1}}\left(b(2b\varepsilon^\gamma)^{2m+1} + \frac{(2\varepsilon^\gamma)^{2m+1}}{2m+1}\right),
\end{aligned}$$

where $c_5(p)$ and $c_6(p)$ are constants independent of m. Here we have used the fact that the series $\sum_{i=0}^\infty \left(\frac{(bT)^i}{i!}\right)^2$ is convergent. We then have

$$\|J_\varepsilon^2\|^2_{C([0,T];L^2(0,1))} \leq c_6(p)\left(b(2b\varepsilon^\gamma)^{2m+1} + \frac{(2\varepsilon^\gamma)^{2m+1}}{2m+1}\right). \tag{79}$$

• Estimate of $\|J_\varepsilon^3 + f_{m,1}^\varepsilon\|_{C([0,T],L^2(0,1))}$ - Using Taylor expansions, for $1-x=\varepsilon z \to 0$, we have

$$\sum_{k=0}^m \varepsilon^k y^k(x,t) = \sum_{k=0}^m \varepsilon^k y^k(1-\varepsilon z,t) = \sum_{k=0}^m \varepsilon^k\left(\sum_{i=0}^{m-k}\frac{1}{i!}\frac{\partial^i y^k}{\partial x^i}(1,t)(-\varepsilon z)^i\right) + R_m^\varepsilon(\xi_z,t),$$

with $\xi_z \in (1-\varepsilon z, 1)$ and

$$R_m^\varepsilon(\xi_z,t) = \varepsilon^{m+1}\sum_{k=0}^m \frac{(-z)^{m-k+1}}{(m-k+1)!}\frac{\partial^{m-k+1}y^k}{\partial x^{m-k+1}}(\xi_z,t).$$

Then we have

$$\sum_{k=0}^m \varepsilon^k Y^k\left(\frac{1-x}{\varepsilon},t\right) - \sum_{k=0}^m \varepsilon^k y^k(x,t) = -R_m^\varepsilon(\xi_z,t) + e^{-Mz}\sum_{k=0}^m \varepsilon^k P^k(z,t). \tag{80}$$

We deduce that

$$\begin{aligned}
&\|J_\varepsilon^3 + f_{m,1}^\varepsilon\|^2_{C([0,T];L^2(0,1))} \\
&= M^2\varepsilon^{-2\gamma}\left\|\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\left(\sum_{k=0}^m \varepsilon^k Q^k\left(z,t\right) - \sum_{k=0}^m \varepsilon^k y^k(1-\varepsilon z,t)\right)\right\|^2_{C([0,T];L^2(0,1))} \\
&\leq M^2\varepsilon^{1-2\gamma}\max_{t\in[0,T]}\int_{\frac{\varepsilon^\gamma}{\varepsilon}}^{\frac{2\varepsilon^\gamma}{\varepsilon}} \left|R_m^\varepsilon(\xi_z,t)\right|^2\,dz.
\end{aligned} \tag{81}$$

We have

$$\begin{aligned}
|R_m^\varepsilon(\xi_z,t)| &\le \varepsilon^{m+1}\sum_{k=0}^{m}\frac{z^{m-k+1}}{(m-k+1)!}\max_{(x,t)\in\overline{Q_T}}\left|\frac{\partial^{m-k+1}y^k}{\partial x^{m-k+1}}(x,t)\right| \\
&\le \varepsilon^{m+1}\sum_{k=0}^{m}\frac{z^{m-k+1}}{(m-k+1)!}\max_{x\ge Mt}\left|\frac{\partial^{m-k+1}y^k}{\partial x^{m-k+1}}(x,t)\right| \\
&\quad + \varepsilon^{m+1}\sum_{k=0}^{m}\frac{z^{m-k+1}}{(m-k+1)!}\max_{x\le Mt}\left|\frac{\partial^{m-k+1}y^k}{\partial x^{m-k+1}}(x,t)\right| \\
&\le c_0(\varepsilon b)^{m+1}\sum_{k=0}^{m}\frac{z^{m-k+1}}{(m-k+1)!}\frac{(Tb)^k}{k!}+c_7(p)\varepsilon^{m+1}\sum_{k=m-p-1}^{m}\frac{z^{m-k+1}}{(m-k+1)!}, \qquad (82)
\end{aligned}$$

where $c_7(p)$ is a constant independent of m. There is $\varepsilon_2>0$ such that for $0<\varepsilon<\varepsilon_2$ we have

$$\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} z^{2(m-k+1)}\,dz \le \int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} z^{2(m+1)}\,dz = \frac{2^{2m+3}}{2m+3}\frac{\varepsilon^{(2m+3)\gamma}}{\varepsilon^{2m+3}}. \qquad (83)$$

Using the Cauchy-Schwarz inequality, inequalities (82), (83), and the convergence of the series $\sum_{k=0}^{+\infty}\left(\frac{(Tb)^k}{k!}\right)^2$, we find that

$$\int_0^{\frac{2\varepsilon^\gamma}{\varepsilon}} |R_m^\varepsilon(\xi_z,t)|^2\,dz \le c_8(p)(2\varepsilon^\gamma)^{2m+3}\left(\frac{(\varepsilon b)^{2m+2}}{\varepsilon^{2m+3}}+\frac{1}{(2m+3)\varepsilon}\right),$$

where $c_8(p)$ is a constant independent of m. Then

$$\|J_\varepsilon^3+f_{m,1}^\varepsilon\|^2_{C([0,T],L^2(0,1))} \le 4M^2c_8(p)\left(b\varepsilon^\gamma(2b\varepsilon^\gamma)^{2m+1}+\frac{(2\varepsilon^\gamma)^{2m+1}}{2m+3}\right). \qquad (84)$$

- Estimate of $\|J_\varepsilon^4+f_{m,2}^\varepsilon\|_{C([0,T],L^2(0,1))}$ - Using (84) we have

$$\|J_\varepsilon^4+f_{m,2}^\varepsilon\|^2_{C([0,T];L^2(0,1))} \le c_9(p)\varepsilon^{2(1-\gamma)}\left(b\varepsilon^\gamma(2b\varepsilon^\gamma)^{2m+1}+\frac{(2\varepsilon^\gamma)^{2m+1}}{2m+3}\right), \qquad (85)$$

where $c_9(p)$ is a constant independent of m.

- Estimate of $\|J_\varepsilon^5+f_{m,3}^\varepsilon\|_{C([0,T],L^2(0,1))}$ - It results from (80) that

$$\begin{aligned}
\varepsilon^{-1}\sum_{k=0}^{m}\varepsilon^kY_z^k(z,t)+\sum_{k=0}^{m}\varepsilon^ky_x^k(1-\varepsilon z,t) &= -\tilde{R}_m^\varepsilon(z,t)+\varepsilon^{-1}e^{-Mz}\sum_{k=0}^{m}\varepsilon^kP_z^k(z,t) \\
&\quad -\varepsilon^{-1}Me^{-Mz}\sum_{k=0}^{m}\varepsilon^kP^k(z,t),
\end{aligned}$$

with

$$\begin{aligned}
\tilde{R}_m^\varepsilon(z,t) &= \varepsilon^{-1}\frac{\partial R_m^\varepsilon}{\partial z}(z,t)=\varepsilon^{-1}\frac{\partial}{\partial z}\sum_{k=0}^{m}\varepsilon^k\int_1^{1-\varepsilon z}\frac{(1-\varepsilon z-s)^{m-k}}{(m-k)!}\frac{\partial^{m-k+1}y^k}{\partial x^{m-k+1}}(s,t)\,ds \\
&= -\varepsilon^m\frac{\partial y^m}{\partial x}(1-\varepsilon z,t)-\sum_{k=0}^{m}\varepsilon^k\int_1^{1-\varepsilon z}\frac{(1-\varepsilon z-s)^{m-k-1}}{(m-k-1)!}\frac{\partial^{m-k+1}y^k}{\partial x^{m-k+1}}(s,t)\,ds.
\end{aligned}$$

We deduce that

$$\|J_\varepsilon^5+f_{m,3}^\varepsilon\|^2_{C([0,T];L^2(0,1))} \le 4\varepsilon^{3-2\gamma}\max_{t\in[0,T]}\int_{\frac{\varepsilon^\gamma}{\varepsilon}}^{\frac{2\varepsilon^\gamma}{\varepsilon}}\left|\tilde{R}_m^\varepsilon(z,t)\right|^2\,dz. \qquad (86)$$

For $1 \geq Mt$ and $\frac{\varepsilon^\gamma}{\varepsilon} \leq z \leq \frac{2\varepsilon^\gamma}{\varepsilon}$ we have by using (68)

$$\begin{aligned}\left|\tilde{R}^\varepsilon_m(z,t)\right| &\leq c_0\varepsilon^m \frac{T^m}{m!}b^{2m+1} + c_0\sum_{k=0}^m \varepsilon^k \frac{T^k}{k!}b^{m+k+1}\int_{1-\varepsilon z}^1 \frac{(s-1+\varepsilon z)^{m-k-1}}{(m-k-1)!}\,ds \\ &\leq c_0\varepsilon^m \frac{T^m}{m!}b^{2m+1} + c_0\varepsilon^m b^{m+1} z^m \sum_{k=0}^m \frac{T^k}{k!}\frac{b^k}{(m-k)!} \\ &\leq c_0 b\frac{(\varepsilon b^2 T)^m}{m!} + c_0 b e^{bT}(2b\varepsilon^\gamma)^m.\end{aligned}$$

For $1 \leq Mt$ and $\frac{\varepsilon^\gamma}{\varepsilon} \leq z \leq \frac{2\varepsilon^\gamma}{\varepsilon}$ we have

$$\begin{aligned}\left|\tilde{R}^\varepsilon_m(z,t)\right| &\leq c_{10}(p)\varepsilon^m + c_{10}(p)\sum_{k=0}^m \varepsilon^k \int_{1-\varepsilon z}^1 \frac{(s-1+\varepsilon z)^{m-k-1}}{(m-k-1)!}\,ds \\ &\leq c_{10}(p)\varepsilon^m + c_{10}(p)\varepsilon^m z^m \sum_{k=0}^m \frac{1}{(m-k)!} \\ &\leq c_{10}(p)\varepsilon^m + c_{10}(p)e(2\varepsilon^\gamma)^m.\end{aligned}$$

where $c_{10}(p)$ is a constant independent of m. Using (86) we then deduce the estimate

$$\begin{aligned}&\|J^5_\varepsilon + f^\varepsilon_{m,3}\|^2_{C([0,T];L^2(0,1))} \\ &\quad \leq 16\varepsilon^{2-\gamma}\left(\left(c_0 b\frac{(\varepsilon b^2T)^m}{m!}\right)^2 + \left(c_0 b e^{bT}(2b\varepsilon^\gamma)^m\right)^2 + \left(c_{10}(p)\varepsilon^m\right)^2 + \left(c_{10}(p)e(2\varepsilon^\gamma)^m\right)^2\right). \qquad (87)\end{aligned}$$

It results from the estimates (75), (79), (84), (85), (87) that we can choose $\varepsilon_0 > 0$ such that, for any fixed $0 < \varepsilon < \varepsilon_0$, $(d^\varepsilon_m)_{m>0}$ tends to 0, as $m \to \infty$. Thanks to (71), $(z^\varepsilon_m)_{m>0}$ tends to 0, as $m \to \infty$.

It remains to study the limite of $(\tilde{\theta}^\varepsilon_m)_m$. We have

$$\tilde{\theta}^\varepsilon_{m0}(x) = a^\varepsilon_m + b^\varepsilon_m,$$

with

$$\begin{aligned}a^\varepsilon_m(x) &= (1-\mathcal{X}_\varepsilon(x))\left(y_0(x) - \sum_{i=0}^m \frac{(-1)^i}{i!}y_0^{(i)}(1)(\varepsilon z)^i\right), \\ b^\varepsilon_m(x) &= (1-\mathcal{X}_\varepsilon(x))\,e^{-Mz}\sum_{i=0}^m \frac{1}{i!}y_0^{(i)}(1)(\varepsilon z)^i, \quad \left(z = \frac{1-x}{\varepsilon}\right).\end{aligned}$$

Using Taylor's expansion, for $1-x = \varepsilon z \to 0$, we have

$$a^\varepsilon_m(x) = (1-\mathcal{X}_\varepsilon(x))\,\frac{(1-x)^{m+1}}{(m+1)!}y_0^{(m+1)}(\zeta_1), \quad \zeta_1 \in (x,1),$$

hence $|a^\varepsilon_m(x)| \leq c_0\frac{b^{m+1}}{(m+1)!}$, then (a^ε_m) converges uniformly in $[0,1]$ to 0. The series $\sum_{i=0}^\infty \frac{y_0^{(i)}(1)}{i!}(1-x)^i$ is uniformly convergent since in $[1-\varepsilon^\gamma, 1]$

$$\sum_{i=0}^m \left|\frac{y_0^{(i)}(1)}{i!}(1-x)^i\right| \leq c_0\sum_{i=0}^m \frac{b^i}{i!}\varepsilon^{i\gamma} \leq c_0\sum_{i=0}^\infty \frac{b^i}{i!}\varepsilon^{i\gamma} = c_0 e^{b\varepsilon^\gamma}.$$

Then $(\tilde{\theta}^\varepsilon_{m0})_m$ converges uniformly in $[0,1]$ to $\tilde{\theta}^\varepsilon_0$ given by

$$\tilde{\theta}^\varepsilon_0(x) = (1-\mathcal{X}_\varepsilon(x))\,e^{-M\frac{1-x}{\varepsilon}}\sum_{i=0}^\infty \frac{y_0^{(i)}(1)}{i!}(1-x)^i, \quad x \in (0,1).$$

Moreover, $\tilde{\theta}_0^\varepsilon$ satisfies an exponential decay property:

$$|\tilde{\theta}_0^\varepsilon(x)| \le c_0 e^{b\varepsilon^\gamma} e^{-2M\frac{\varepsilon^\gamma}{\varepsilon}}, \quad \forall x \in (0,1).$$

Consider now the function $f_{m,1}^\varepsilon$. We have

$$\sum_{k=0}^m \varepsilon^k \left| P^k\left(\frac{1-x}{\varepsilon}, t\right)\right| \le \sum_{k=0}^m \varepsilon^k \left(\sum_{i=0}^k \left|\frac{\partial y^{k-i}}{\partial x^i}(1,t)\right| \frac{z^i}{i!}\right), \quad (x,t) \in Q_T,\ z = \frac{1-x}{\varepsilon},$$

then, for $1 - 2\varepsilon^\gamma \le x \le 1 - \varepsilon^\gamma$,

$$\begin{aligned}
\sum_{k=0}^m \varepsilon^k \left| P^k\left(\frac{1-x}{\varepsilon}, t\right)\right| &\le c_0 \sum_{k=0}^m \varepsilon^k \left(\sum_{i=0}^k \frac{T^{k-i}|y_0^{(2k-i)}(1-Mt)|}{(k-i)!}\frac{z^i}{i!}\right) + c_{11}(p)\sum_{k=0}^m \varepsilon^k \left(\sum_{i=0}^k \frac{z^i}{i!}\right)\\
&\le c_0 \sum_{k=0}^m b^{2k}T^k\varepsilon^k \left(\sum_{i=0}^k \frac{T^{-i}}{(k-i)!}\frac{b^{-i}z^i}{i!}\right) + c_{11}(p)\sum_{k=0}^m \varepsilon^k \left(\sum_{i=0}^k \frac{z^i}{i!}\right)\\
&\le c_0 \sum_{k=0}^m b^{2k}T^k\varepsilon^k \left(\sum_{i=0}^k \frac{T^{-i}}{(k-i)!}\frac{b^{-i}}{i!}\frac{2^i\varepsilon^{i\gamma}}{\varepsilon^i}\right) + c_{11}(p)\sum_{k=0}^m \varepsilon^k \left(\sum_{i=0}^k \frac{1}{i!}\frac{2^i\varepsilon^{i\gamma}}{\varepsilon^i}\right)\\
&\le c_0 \sum_{k=0}^m b^{2k}T^k\varepsilon^{k\gamma} \left(\sum_{i=0}^k \frac{T^{-i}}{(k-i)!}\frac{b^{-i}}{i!}2^i\right) + c_{11}(p)\sum_{k=0}^m \varepsilon^{k\gamma} \left(\sum_{i=0}^k \frac{2^i}{i!}\right)\\
&\le \frac{c_0 e^{\frac{2}{bT}}}{1 - b^2T\varepsilon^\gamma} + c_{11}(p)\frac{e^2}{1-\varepsilon^\gamma},
\end{aligned}$$

for $0 < \varepsilon^\gamma < \min\left(\frac{1}{b^2T}, 1\right)$, where $c_{11}(p)$ is a constant independent on m. Then, for $0 < \varepsilon^\gamma < \min\left(\frac{1}{b^2T}, 1\right)$, the series $\sum_{k=0}^\infty \varepsilon^k P^k\left(\frac{1-x}{\varepsilon}, t\right)$ is uniformly convergent in $\overline{Q_T}$, therefore $(f_{m,1}^\varepsilon)_m$ converges uniformly in $\overline{Q_T}$ to a function f_1^ε given by

$$f_1^\varepsilon(x,t) = M\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{-\gamma}e^{-M\frac{1-x}{\varepsilon}}\sum_{k=0}^\infty \varepsilon^k P^k\left(\frac{1-x}{\varepsilon}, t\right), \quad (x,t) \in Q_T.$$

Moreover, f_1^ε has an exponential decay property:

$$|f_1^\varepsilon(x,t)| \le M\left|\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\right| e^{-M\frac{\varepsilon^\gamma}{\varepsilon}}\left(\frac{c_0 e^{\frac{2}{bT}}}{1-b^2T\varepsilon^\gamma} + c_{11}(p)\frac{e^2}{1-\varepsilon^\gamma}\right), \quad (x,t) \in Q_T.$$

Clearly, $(f_{m,2}^\varepsilon)_m$ converges uniformly in $\overline{Q_T}$ to a function f_2^ε satisfying a property of exponential decay. Similarly we show that $(f_{m,3}^\varepsilon)_m$ converges uniformly in $\overline{Q_T}$ to a function f_3^ε satisfying a property of exponential decay. Thus $(f_m^\varepsilon)_m$ converges uniformly in $\overline{Q_T}$ to a function $f^\varepsilon = f_1^\varepsilon + f_2^\varepsilon + f_3^\varepsilon$ satisfying a property of exponential decay.

Let $\tilde{\theta}^\varepsilon$ be the solution of the problem

$$\begin{cases} L_\varepsilon(\tilde{\theta}^\varepsilon) = f^\varepsilon, & (x,t) \in Q_T),\\ \tilde{\theta}^\varepsilon(0,t) = \tilde{\theta}^\varepsilon(1,t) = 0, & t \in (0,T),\\ \tilde{\theta}^\varepsilon(x,0) = \tilde{\theta}_0^\varepsilon(x), & x \in (0,1). \end{cases} \tag{88}$$

We easily deduce from (88) that $\tilde{\theta}^\varepsilon$ belongs to $C^\infty(\overline{Q_T})$ and has an exponential decay property:

$$|\tilde{\theta}^\varepsilon(x,t)| \le c_{12}(p)e^{-M\frac{\varepsilon^\gamma}{\varepsilon}}, \quad (x,t) \in Q_T,$$

where $c_{12}(p)$ is a constant independent on m. Then from (70) and (88) we deduce that

$$\|\tilde{\theta}_m^\varepsilon(\cdot,t) - \tilde{\theta}^\varepsilon(\cdot,t)\|_{L^2(0,1)}^2 \le \left(\|f_m^\varepsilon - f^\varepsilon\|_{L^2(Q_T)}^2 + \|\tilde{\theta}_{m0}^\varepsilon - \tilde{\theta}_0^\varepsilon\|_{L^2(0,1)}\right)e^t, \quad \forall t \in (0,T),$$

which implies that $(\tilde{\theta}_m^\varepsilon)_m$ converges in $C([0,T]; L^2(0,1))$ to $\tilde{\theta}^\varepsilon$. This completes the proof of the theorem. $\square$

Remark 3 *We can actually weaken the condition (68) by the following one : the initial condition y_0 belongs to $C^\infty([0,1])$ and there are $c_0, b \in \mathbb{R}$ such that*

$$\|y_0^{(2(m+1))}\|_{L^2(0,1)} \leq c_0\,(m+1)!\; b^m, \quad \forall m \in \mathbb{N}. \tag{89}$$

Let us consider the term J_ε^1 on $Q_T^- = \{(x,t) \in Q_T, x \geq Mt\}$. From (72), we have

$$\begin{aligned}
\|J_1^\varepsilon\|^2_{L^2(Q_T^-)} &\leq \frac{\varepsilon^{2(m+1)}}{(m!)^2}\int_0^T t^{2m}\int_{Mt}^1 |y_0^{2m+2)}(x-Mt)|^2 dxdt \\
&\leq \frac{\varepsilon^{2(m+1)}}{(m!)^2}\frac{T^{2m+1}}{2m+1}\|y_0^{(2m+2)}\|^2_{L^2(0,1)} \\
&\leq (c_0)^2\frac{\varepsilon^{2(m+1)}}{(m!)^2}\frac{T^{2m+1}}{2m+1}((m+1)!)^2 b^{2m} \\
&\leq (c_0)^2\varepsilon^2 T\frac{(m+1)^2}{2m+1}(T\varepsilon b)^{2m} \to 0 \;\; \textit{as} \;\; m\to\infty \;\; \textit{if} \;\; T\varepsilon b < 1.
\end{aligned} \tag{90}$$

The other terms J_ε^i, $i = 2, \cdots, 5$ can be treated in a similar way.

2.5 Asymptotic approximation of the adjoint solution φ^ε

Let us consider the adjoint problem

$$\begin{cases} -\varphi_t^\varepsilon - \varepsilon\varphi_{xx}^\varepsilon - M\varphi_x^\varepsilon = 0, & (x,t) \in Q_T, \\ \varphi^\varepsilon(0,t) = \varphi^\varepsilon(1,t) = 0, & t \in (0,T), \\ \varphi^\varepsilon(x,T) = \varphi_T^\varepsilon(x), & x \in (0,1), \end{cases} \tag{91}$$

where φ_T^ε is a function of the form $\varphi_T^\varepsilon = \sum_{k=0}^m \varepsilon^k \varphi_T^k$, the functions φ_T^0, $\varphi_T^1, \cdots, \varphi_T^m$ being given. We assume $M > 0$, the case $M < 0$ can be treated similarly. We construct an asymptotic approximation of the solution φ^ε of (91) by using the matched asymptotic expansion method.

To get the outer expansion $\sum_{k=0}^m \varepsilon^k \varphi^k(x,t)$ of φ^ε we repeat again the procedure performed for the direct solution y^ε. From equation (91) we have

$$\begin{aligned}
\varepsilon^0 : &\quad \varphi_t^0 + M\varphi_x^0 = 0, \\
\varepsilon^k : &\quad \varphi_t^k + M\varphi_x^k = -\varphi_{xx}^{k-1}, \quad 1 \leq k \leq m.
\end{aligned}$$

Taking the initial and boundary conditions into account we define φ^0 and φ^k $(1 \leq k \leq m)$ as functions satisfying the transport equations, respectively,

$$\begin{cases} \varphi_t^0 + M\varphi_x^0 = 0, & (x,t) \in Q_T, \\ \varphi^0(1,t) = 0, & t \in (0,T), \\ \varphi^0(x,T) = \varphi_T^0(x), & x \in (0,1), \end{cases} \tag{92}$$

and

$$\begin{cases} \varphi_t^k + M\varphi_x^k = -\varphi_{xx}^{k-1}, & (x,t) \in Q_T, \\ \varphi^k(1,t) = 0, & t \in (0,T), \\ \varphi^k(x,T) = \varphi_T^k(x), & x \in (0,1). \end{cases} \tag{93}$$

The solution of (92) is given by

$$\varphi^0(x,t) = \begin{cases} 0, & x > 1 + M(t-T), \\ \varphi_T^0\big(x + M(T-t)\big), & x < 1 + M(t-T). \end{cases} \tag{94}$$

Using the method of characteristics we find that, for any $1 \leq k \leq m$,

$$\varphi^k(x,t) = \begin{cases} \displaystyle\int_t^{t+(1-x)/M} \varphi_{xx}^{k-1}\big(x+M(s-t),s\big)ds, & x > 1+M(t-T), \\ \displaystyle\varphi_T^k\big(x+M(T-t)\big) + \int_t^T \varphi_{xx}^{k-1}\big(x+M(s-t),s\big)ds, & x < 1+M(t-T). \end{cases} \tag{95}$$

The inner expansion is given by

$$\sum_{k=0}^{m} \varepsilon^k \Phi^k(z,t), \quad z = \frac{x}{\varepsilon} \in (0,\varepsilon^{-1}),\, t \in (0,T),$$

with functions Φ^0 and Φ^k $(1 \leq k \leq m)$ satisfying the equations, respectively,

$$\begin{aligned} &\Phi_{zz}^0(z,t) + M\Phi_z^0(z,t) = 0, \\ &\Phi_{zz}^k(z,t) + M\Phi_z^k(z,t) = -\Phi_t^k(z,t). \end{aligned}$$

We define Φ^0 as a solution of

$$\begin{cases} \Phi_{zz}^0(z,t) + M\Phi_z^0(z,t) = 0, & (z,t) \in (0,+\infty)\times(0,T), \\ \Phi^0(0,t) = 0, & t \in (0,T), \\ \lim\limits_{z\to+\infty} \Phi^0(z,t) = \lim\limits_{x\to 0} \varphi^0(x,t), & t \in (0,T). \end{cases} \tag{96}$$

The solution of (96) reads

$$\Phi^0(z,t) = \varphi^0(0,t)\left(1 - e^{-Mz}\right), \quad (z,t) \in (0,+\infty)\times(0,T). \tag{97}$$

For $1 \leq k \leq m$, the function Φ^k is defined iteratively as a solution of

$$\begin{cases} \Phi_{zz}^k(z,t) + M\Phi_z^k(z,t) = -\Phi_t^{k-1}(z,t), & (z,t) \in (0,+\infty)\times(0,T), \\ \Phi^k(0,t) = 0, & t \in (0,T), \\ \lim\limits_{z\to+\infty} [\Phi^k(z,t) - S^k(z,t)] = 0, & t \in (0,T), \end{cases} \tag{98}$$

where

$$S^k(z,t) = \sum_{i=0}^{k} \frac{1}{i!} \frac{\partial^i \varphi^{k-i}}{\partial x^i}(0,t) z^i.$$

Arguing as in Lemma 2.5 one can verify that the solution of problem (98) reads

$$\Phi^k(z,t) = S^k(z,t) + e^{-Mz} R^k(z,t), \quad (z,t) \in (0,+\infty)\times(0,t), \tag{99}$$

where

$$R^k(z,t) = \sum_{i=0}^{k} \frac{(-1)^{i+1}}{i!} \frac{\partial^i \varphi^{k-i}}{\partial x^i}(0,t) z^i.$$

Let $\mathcal{X} : \mathbb{R} \to [0,1]$ denote a C^∞ cut-off function satisfying (19). We define, for $\gamma \in (0,1)$, the function

$$\mathcal{X}_\varepsilon(x) = \mathcal{X}\left(\frac{x}{\varepsilon^\gamma}\right),$$

then introduce the function

$$\psi_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^{m} \varepsilon^k \varphi^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^{m} \varepsilon^k \Phi^k\left(\frac{x}{\varepsilon}, t\right), \tag{100}$$

defined to be an asymptotic approximation at order m of the solution φ^ε of (91). To justify all the computations we will perform we need some regularity assumptions on the data φ^0 and φ^k, $1 \leq k \leq m$. We have the following result.

Lemma 2.11 *Assume that, for any $0 \le k \le m$, $\varphi_T^k \in C^{2(m-k)+1}[0,T]$, and the following $C^{2(m-k)+1}$-matching conditions are satisfied, respectively,*

$$(\varphi_T^k)^{(p)}(1) = 0, \quad 0 \le p \le 2(m-k)+1. \tag{101}$$

Then the function φ^k belongs to $C^{2(m-k)+1}(\overline{Q_T})$.

A straightforward calculation gives

$$L_\varepsilon^\star(\psi_m^\varepsilon)(x,t) = \sum_{i=1}^{5} E_\varepsilon^i(x,t),$$

with

$$\begin{aligned}
E_\varepsilon^1(x,t) &= -\varepsilon^{m+1}\varphi_{xx}^m(x,t)\mathcal{X}_\varepsilon(x),\\
E_\varepsilon^2(x,t) &= -\varepsilon^m(1-\mathcal{X}_\varepsilon(x))\Phi_t^m\left(\frac{x}{\varepsilon},t\right),\\
E_\varepsilon^3(x,t) &= M\mathcal{X}'\left(\frac{x}{\varepsilon^\gamma}\right)\varepsilon^{-\gamma}\left(\sum_{k=0}^m \varepsilon^k\Phi^k\left(\frac{x}{\varepsilon},t\right) - \sum_{k=0}^m \varepsilon^k\varphi^k(x,t)\right),\\
E_\varepsilon^4(x,t) &= \mathcal{X}''\left(\frac{x}{\varepsilon^\gamma}\right)\varepsilon^{1-2\gamma}\left(\sum_{k=0}^m \varepsilon^k\Phi^k\left(\frac{x}{\varepsilon},t\right) - \sum_{k=0}^m \varepsilon^k\varphi^k(x,t)\right),\\
E_\varepsilon^5(x,t) &= 2\mathcal{X}'\left(\frac{x}{\varepsilon^\gamma}\right)\varepsilon^{1-\gamma}\left(\varepsilon^{-1}\sum_{k=0}^m \varepsilon^k\Phi_z^k\left(\frac{x}{\varepsilon},t\right) - \sum_{k=0}^m \varepsilon^k\varphi_x^k(x,t)\right).
\end{aligned}$$

We have the analogue of Lemma 2.7.

Lemma 2.12 *Assume that the assumptions of Lemma 2.11 hold true. Let ψ_m^ε be the function defined by* (100)*. Then there is a constant c_m independent of ε such that*

$$\|L_\varepsilon^\star(\psi_m^\varepsilon)\|_{C([0,T];L^2(0,1))} \le c_m\varepsilon^{\frac{(2m+1)\gamma}{2}}. \tag{102}$$

Using Lemma 2.12 we can argue as in the proof of Theorem 2.1 to establish the following result.

Theorem 2.4 *Assume that the assumptions of Lemma 2.11 hold true. Let φ^ε be the solution of problem* (91) *and let ψ_m^ε be the function defined by* (100)*. Then there is a constant c_m independent of ε such that*

$$\|\varphi^\varepsilon - \psi_m^\varepsilon\|_{C([0,T];L^2(0,1))} \le c_m\varepsilon^{\frac{2m+1}{2}\gamma}. \tag{103}$$

3 Approximate controllability results

We may use the previous asymptotic analysis to state ε-approximate controllability results. Preliminary, let us prove the following decay property of the solution y^ε in the uncontrolled case.

Proposition 3.1 *Let y^ε be the solution of (1) with $v^\varepsilon \equiv 0$ and $L = 1$. Let any $\alpha \in (0,1)$. Then, the solution y^ε satisfies the following estimate*

$$\|y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \le \|y^\varepsilon(\cdot,0)\|_{L^2(0,1)}e^{-\frac{M\alpha^2}{4\varepsilon(1-\alpha)}}, \quad \forall t \ge \frac{1}{M(1-\alpha)}.$$

Proof- We define $z^\varepsilon(x,t) = e^{\frac{-M\alpha x}{2\varepsilon}}y^\varepsilon(x,t)$ and then check that

$$L_\varepsilon(y^\varepsilon) = e^{\frac{M\alpha x}{2\varepsilon}}\left(z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + M(1-\alpha)z_x^\varepsilon - \frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)z^\varepsilon\right), \quad \forall (x,t) \in Q_T. \tag{104}$$

Consequently, z^ε is a solution of

$$\begin{cases} z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + M(1-\alpha)z_x^\varepsilon - \frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)z^\varepsilon = 0 & \text{in} \quad Q_T, \\ z^\varepsilon(0,\cdot) = z^\varepsilon(1,\cdot) = 0 & \text{on} \quad (0,T), \\ z^\varepsilon(\cdot,0) = e^{\frac{-M\alpha x}{2\varepsilon}} y_0^\varepsilon & \text{in} \quad (0,1). \end{cases}$$

Multiplying the main equation by z^ε and integrating over $(0,1)$ then leads to

$$\frac{d}{dt}\|z^\varepsilon(\cdot,t)\|^2 + 2\varepsilon\|z_x^\varepsilon(\cdot,t)\|^2_{L^2(0,1)} \leq \frac{M^2}{2\varepsilon}(\alpha^2-2\alpha)\|z^\varepsilon(\cdot,t)\|^2_{L^2(0,1)}$$

and then to the estimate $\|z^\varepsilon(\cdot,t)\|_{L^2(0,1)} \leq \|z^\varepsilon(\cdot,0)\|_{L^2(0,1)} e^{\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)t}$, equivalently, to

$$\|e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \leq \|e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,0)\|_{L^2(0,1)} e^{\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)t}.$$

Consequently,

$$\begin{aligned} \|y^\varepsilon(\cdot,t)\|_{L^2(0,1)} &= \|e^{\frac{M\alpha x}{2\varepsilon}} e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \leq \|e^{\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)} \|e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,t)\|_{L^2(0,1)} \\ &\leq \|e^{\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)} \|e^{-\frac{M\alpha x}{2\varepsilon}} y^\varepsilon(\cdot,0)\|_{L^2(0,1)} e^{\frac{M^2}{4\varepsilon}(\alpha^2-2\alpha)t} \\ &\leq \|y^\varepsilon(\cdot,0)\|_{L^2(0,1)} e^{\frac{M\alpha}{2\varepsilon}\left(1-Mt+\frac{M\alpha t}{2}\right)} \end{aligned}$$

using that (recall that $\alpha > 0$) $\|e^{\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)} = e^{\frac{M\alpha}{2\varepsilon}}$ and $\|e^{-\frac{M\alpha x}{2\varepsilon}}\|_{L^\infty(0,1)} = 1$. Let now $t \geq \frac{1}{M(1-\alpha)} > \frac{1}{M}$ so that $(1 - Mt + \frac{M\alpha t}{2}) \leq -\frac{\alpha}{2(1-\alpha)}$ and $\frac{M\alpha}{2}(1 - Mt + \frac{M\alpha t}{2}) \leq -\frac{M\alpha^2}{4(1-\alpha)}$. The result follows. □

Consequently, as soon as the controllability time T is strictly larger than $1/M$, the L^2-norm of the free solution at time T is exponentially small with respect to ε. This is in agreement with the weak limit given by (2) but show how the related controllability problem is singular.

Remark that the solution y^ε belongs to $C^\infty([0,1]\times[\eta,T])$ for all $\eta > 0$. The solution y^ε belongs to $C^\infty([0,1]\times[0,T])$ if in addition the initial data satisfies regularity and compatibility assumptions (for the heat equation, $y_0^\varepsilon \in H^k, \forall k$ and $y_0^\varepsilon = (y_0^\varepsilon)^{(2j)} = 0$ at $x = 0,1$, see Theorem 10.2 in [1]). On the other hand, thank to the compatibility conditions of Lemma 2.6, the approximation w_m^ε is continuous from $t = 0$.

The asymptotic analysis performed in the previous section leads for $T > 1/M$ to the following approximate controllability result.

PROPOSITION 3.2 *Let $m \in \mathbb{N}$, $T > \frac{1}{M}$ and $a \in]0, T - \frac{1}{M}[$. Assume that the assumptions on the initial condition y_0 and functions v^k, $0 \leq k \leq m$ of Lemma 2.6 hold true. Assume moreover that*

$$v^k(t) = 0, \quad 0 \leq k \leq m, \forall t \in [a,T]. \tag{105}$$

Then, the solution y^ε of problem (7) satisfies the following property

$$\|y^\varepsilon(\cdot,T)\|_{L^2(0,1)} \leq c_m \varepsilon^{\frac{(2m+1)\gamma}{2}}, \quad \forall\gamma \in (0,1)$$

for some constant $c_m > 0$ independent of ε.

In other words, the function $v^\varepsilon \in C([0,T])$ defined by $v^\varepsilon := \sum_{k=0}^m \varepsilon^k v^k$ is an approximate null control for (1): for any $\eta > 0$, there exists ε_0, such that for any $\varepsilon \in (0,\varepsilon_0)$, $\|y^\varepsilon(\cdot,T)\|_{L^2(0,1)} \leq \eta$.

PROOF. We first check by induction that the function y^k, $0 \leq k \leq m$ given by (10) and (11) vanishes at time T. From (10) and the assumption (105), $y^0(x,t) = 0$ on the set

$$S_a := \{(x,t) \in (0,1)\times(0,T), tM - x \geq aM\}$$

which contains the set $S_{T-1/M}$ and the set $\{0,1\}\times\{T\}$. Assume now that $y^{k-1}(x,t)=0$ on S_a, for some $k\geq 1$. (48) implies that, for all $(x,t)\in S_a$

$$y^k(x,t)=v^k\left(t-\frac{x}{M}\right)+\int_{t-x/M}^{t} y_{xx}^{k-1}\left(x+M(s-t),s\right)ds.$$

From (105), the first term vanishes because $t-\frac{x}{M}\geq a$ for all $(x,t)\in S_a$. Moreover, for $(x,t)\in S_a$, the segment $[x+M(s-t),s]$ for $s\in[t-x/M,t]\subset[a,T]$ belongs to S_a. Consequently, the second term vanishes as well and $y^k(x,t)=0$, $0\leq k\leq m$ for all $(x,t)\in S_a$. In particular $y^k(x,T)=0$, $0\leq k\leq m$ for all $x\in[0,1]$. Then, the relation (41) implies that the function Y^k satisfies for all $0\leq k\leq m$, $Y^k(z,T)=0$ for all $z\in[0,\infty)$. Consequently, the function w_m^ε defined by (45) satisfies $w_m^\varepsilon(\cdot,T)=0$ on $[0,1]$. The result follows from the inequality $\|y^\varepsilon(\cdot,T)\|_{L^2(0,1)}\leq\|y^\varepsilon(\cdot,T)-w_m^\varepsilon(\cdot,T)\|_{L^2(0,1)}$ and Proposition 2.2. Figure 2-left illustrates this result. $\square$

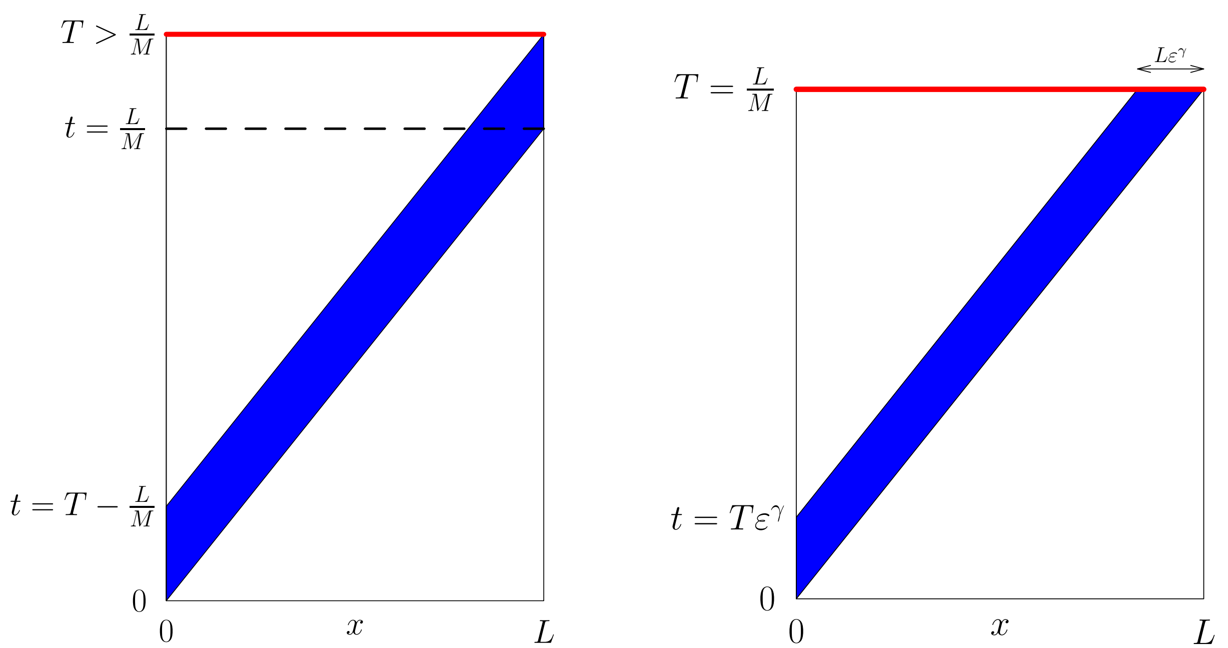

Figure 2: Influence zone of the control v^ε (as $\varepsilon\to 0$) in Q_T delimited by the characteristic line $Mt-x=0$ for $T\geq\frac{L}{M}$ and $T=\frac{L}{M}$.

Remark 4 *For ε small enough but fixed, we can not pass to the limit as $m\to\infty$ in order to get a null controllability result. This is due to the fact that the constant c depends on the parameter m, and more precisely on the $(2(m-k)+1)$ first derivatives of the function v^k, $0\leq k\leq m$. Let us consider simply the initial condition $y_0(x)=1$ for which $y_0(0)=1$ and $(y_0)^{(p)}(0)=0$ for all $p>0$. We then check that the functions v^k, $k\geq 0$ defined as follows*

$$v^0(t)=\mathcal{X}(t),\quad v^k(t)=0,\quad k>0$$

with $\mathcal{X}=\{f\in C^\infty[0,T],f(0)=1,f(a)=0,f^{(p)}(0)=f^{(p)}(a)=0,p\in\mathbb{N}^\star\}$, $a\in[0,T]$ satisfy the matching conditions of Lemma 2.6. As before, if $a\in]0,T-1/M[$, then $\|w_m^\varepsilon(\cdot,T)\|_{L^2(0,1)}=0$ for all $m\in\mathbb{N}$. If for such fonctions v^k, $0\leq k\leq m$, the term $c_m\varepsilon^{\frac{(2m+1)\gamma}{2}}$ goes to zero as $m\to\infty$, this implies that the function $v^\varepsilon=\sum_{k=0}^\infty\varepsilon^k v^k$ is a null control for y^ε as time T. However, v^ε is here simply $v^\varepsilon=v^0$, which is not a null control for y^ε, $\varepsilon>0$ fixed !

On the other hand, according to Theorem 2.3, if there exists $p_0\in\mathbb{N}$ such that $(v^0)^p(t)=0$, $t\in[0,T]$ $\forall p\geq p_0$ (take for instance $v^0(t)=1$ satisfying the matching conditions), then the term $c_m\varepsilon^{\frac{(2m+1)\gamma}{2}}$ goes to zero as $m\to\infty$ (as well as the term $\|y^\varepsilon-w_m^\varepsilon\|_{C([0,T],L^2(0,1))}$) but $\|w_m^\varepsilon(\cdot,T)\|_{L^2(0,1))}\neq 0$ for all m.

Remark 5 *The limit case $T = 1/M$ can be considered as well but requires explicit formula. The function $w_m^\varepsilon(\cdot, T)$ is no longer equal to zero in this case. Let us consider for simplicity the case $m = 0$ for which*

$$w_0^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) y^0(x,t) + (1-\mathcal{X}_\varepsilon(x))\, Y^0\left(\frac{1-x}{\varepsilon}, t\right), \quad (x,t) \in Q_T$$

so that

$$w_0^\varepsilon(x,T) = \mathcal{X}_\varepsilon(x) y^0(x,T) + (1-\mathcal{X}_\varepsilon(x))\, Y^0\left(\frac{1-x}{\varepsilon}, T\right), \quad x \in (0,1).$$

First, (15) leads to $Y^0\left(\frac{1-x}{\varepsilon}, T\right) = y_0(0)(1 - e^{-\frac{M(1-x)}{\varepsilon}})$. Therefore,

$$\left\|(1-\mathcal{X}_\varepsilon(x))\, Y^0\left(\frac{1-x}{\varepsilon}, T\right)\right\|^2_{L^2(0,1)} = (y_0(0))^2 \int_0^1 (1-\mathcal{X}_\varepsilon(x))^2 (1-e^{-Mz})^2 dx.$$

Writing that $(1 \pm e^{-Mz}) \le 2$ and that $\|\,(1-\mathcal{X}_\varepsilon(x))\,(1-x)^p\|_{L^2(0,1)} = \mathcal{O}(\varepsilon^{(2p+1)\gamma/2}), p \ge 0$, we obtain that

$$\left\|(1-\mathcal{X}_\varepsilon(x))\, Y^0\left(\frac{1-x}{\varepsilon}, T\right)\right\|_{L^2(0,1)} = |y_0(0)|\, \varepsilon^{\gamma/2}.$$

Moreover, from (10), we obtain, for all $x \in (0,1)$, that

$$y^0(x,T) = v^0(\overline{x}), \quad \overline{x} := \frac{1-x}{M} = T(1-x)$$

and we may easily define a function v^0 such that the norm $\|\mathcal{X}_\varepsilon(x) y^0(x,T)\|_{L^2(0,1)}$ be equal to zero. Actually, since the function $\mathcal{X}_\varepsilon$ is supported in $[0, 1-\varepsilon^\gamma]$, it suffices to take a function v^0 such that $v^0(\overline{x}) = 0$ for $x \in [0, 1-\varepsilon^\gamma]$, i.e. supported in $[0, \varepsilon^\gamma T]$ (see Figure 2-right). Consequently, such control v^0 leads to

$$\|w_0^\varepsilon(\cdot, T)\|_{L^2(0,1)} \le |y_0(0)|\, \varepsilon^{\gamma/2}.$$

It remains to evaluate the term $\|y^\varepsilon(\cdot,T) - w_0^\varepsilon(\cdot,T)\|_{L^2(0,1)}$, equivalently evaluate the term $\|L^\varepsilon w_0^\varepsilon\|_{C([0,T],L^2(0,1))}$. In order to satisfy the matching conditions of Lemma 2.1, we define v^0 as follows

$$v^0(t) = \sum_{p=0}^{1} (-1)^p \frac{(tM)^p}{p!} (y_0(0))^{(p)} \mathcal{X}(t), \tag{106}$$

for any $C^1([0,a],[0,1])$-function $\mathcal{X}$ such that $\mathcal{X}(0) = 1, (\mathcal{X})^k(0) = 0, \mathcal{X}(a) = 0, (\mathcal{X})^k(a) = 0, k = 0,1$ with $a \in]0, \varepsilon^\gamma T]$. The function v^0 (and in particular the derivatives) depends on ε here and so the constant c_m in (54).

Let us evaluate the first term J_ε^1 of $L_\varepsilon w_0^\varepsilon$ (see (55)), restricted to $Q_T^+ := \{(x,t) \in Q_T, x - Mt \le 0\}$, in function of the support $(0,a)$ of v^0: from $J_\varepsilon^1 = -\varepsilon y_{xx}^0 = -\frac{\varepsilon}{M^2}(v^0)^{(2)}(t - x/M)$, we have

$$\begin{aligned}
\|J_\varepsilon^1\|^2_{L^2(Q_T^+)} =& \frac{\varepsilon^2}{M^4} \int_0^1 \int_{\frac{x}{M}}^T \left((v^0)^{(2)}(t - x/M)\right)^2 dt\, dx \\
=& \frac{\varepsilon^2}{M^4} \int_0^1 \int_0^{T-x/M} \left((v^0)^{(2)}(t)\right)^2 dt\, dx = \frac{\varepsilon^2}{M^4} \int_0^1 \int_0^{max(T-x/M,a)} \left((v^0)^{(2)}(t)\right)^2 dt\, dx \\
\le& \frac{\varepsilon^2}{M^4} \int_0^1 \int_0^a \left((v^0)^{(2)}(t)\right)^2 dt\, dx = \frac{1}{M^4}\left(\varepsilon \|(v^0)^{(2)}\|_{L^2(0,a)}\right)^2.
\end{aligned} \tag{107}$$

Let us consider the polynomial of order 3 given by $\mathcal{X}(t) = 1 - 3(t/a)^2 + 2(t/a)^3$ so that $\mathcal{X}(0) = 1$ and $\mathcal{X}'(0) = \mathcal{X}(a) = \mathcal{X}'(a) = 0$ for all $a \ne 0$. Moreover, to simplify even more the computation, let assume that $y_0^{(1)}(0) = 0$ so that the control v^0 is simply given by $v^0(t) = y_0(0)\mathcal{X}(t) 1_{[0,a]}(t)$ leading to $\|(v^0)^{(2)}\|_{L^2(0,a)} = \frac{12|y_0(0)|}{a^{3/2}}$ and then (from (107))

$$\|J_\varepsilon^1\|_{L^2(Q_T^+)} \le \frac{12\varepsilon}{a^{3/2}M^2} |y_0(0)|.$$

We are therefore looking for $a \leq T\varepsilon^\gamma$ such that $\varepsilon^2/a^3 \to 0$ as $\varepsilon \to 0$. We take $a = T\varepsilon^{\gamma'}$. This requires $\gamma' \in [\gamma, 2/3]$ and then $\|J_\varepsilon^1\|_{L^2(Q_T^+)} \leq \frac{12|y_0(0)|}{M^2}\varepsilon^{1-3\gamma'/2} \leq \frac{12|y_0(0)|}{T^{3/2}M^2}\varepsilon^{1-3\gamma/2}$.

We can proceed in a similar way with the other terms in (55) and determine a rate $\tau = \tau(\gamma)$ such that $\|L_\varepsilon(w_0^\varepsilon)\|_{L^2(Q_T)} \leq c\varepsilon^\tau$ and then $\|y^\varepsilon(\cdot, T) - w_0^\varepsilon(\cdot, T)\|_{L^2(0,1)} \leq c\varepsilon^\tau$. This allows to conclude that there exists a control function $v^0 \in C^1([0, T\varepsilon^\gamma])$ such that the solution of (7) with $v^\varepsilon = v^0$ satisfies

$$\|y^\varepsilon(\cdot, T)\|_{L^2(0,1)} \leq |y_0(0)|(c\varepsilon^{\tau(\gamma)} + \varepsilon^{\gamma/2}), \tag{108}$$

with $\gamma < 2/3$ (instead of $\gamma < 1$ in Proposition 3.2). This stronger condition shows how the convergence is affected in the limit case $T = 1/M$. Nevertheless, after tedious computations, we may extend this construction of v^0 to any order k and improve the rate in the estimate (108). This may allow to obtain a better estimate that in the uncontrolled case discussed in Proposition 3.1. Remark that in the uncontrolled case, the norm $\|y^\varepsilon(\cdot, T)\|_{L^2(0,1)}$ is a priori not exponentially small for $T = 1/M$.

4 The case of initial condition y_0^ε dependent on ε

The asymptotic analysis performed in Section 2 requires *a priori* more care if the initial condition depends on the parameter ε. Due to the compatibility conditions of Lemma 2.6, the control functions v^k may then depend on ε (at least in the neighborhood of $t = 0$) and so the constant c_m in (67).

In view of the initial condition defined in (4) (highlighted in [2, 10]), we have in particular in mind the initial data of the form $y_0^\varepsilon(x) = c_\varepsilon f(x)e^{\frac{M\alpha}{2\varepsilon}}$, with $\alpha < 0$, $c_\varepsilon \in \mathbb{R}$. Such initial data get concentrated as $\varepsilon \to 0$ at the point $x = 0$ (resp. $x = 1$) for $M > 0$ (resp. $M < 0$). Precisely, let us consider the case of the $C^\infty([0,1])$ initial data (4) (with $L = 1$):

$$y_0^\varepsilon(x) := K_\varepsilon \sin(\pi x)e^{-\frac{Mx}{2\epsilon}}, \quad K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2}) \tag{109}$$

such that $\|y_0^\varepsilon\|_{L^2(0,1)} = 1$. Taking $m = 0$ in Lemma 2.7, the function $L_\varepsilon(w_0^\varepsilon)$ involves the term $-\varepsilon y_{xx}^0(x,t)\mathcal{X}_\varepsilon(x)$ where y^0 is given by (10). In particular, for points below the characteristic, that is in the set $Q_T^- := \{(x,t) \in Q_T, x > Mt\}$, we obtain $y^0(x,t) = y_0^\varepsilon(x - Mt)$; this leads, after some computations, to (writing that $\mathcal{X}_\varepsilon = 1$ on $(0, 1 - 2\varepsilon^\gamma)$)

$$\begin{aligned} \varepsilon\|y_{xx}^0\mathcal{X}_\varepsilon\|_{L^2(Q_T^-)} &\geq \varepsilon K_\varepsilon\left(\int_0^{1-2\varepsilon^\gamma}\int_0^{x/M}\left((\sin(\pi(x - Mt))e^{-\frac{M(x-Mt)}{2\varepsilon}})_{xx}\right)^2 dtdx\right)^{1/2} \\ &= \varepsilon K_\varepsilon\mathcal{O}(1) = \mathcal{O}(\varepsilon^{-1/2}). \end{aligned} \tag{110}$$

Consequently, w_0^ε can not be a convergent approximation of y^ε as $\varepsilon \to 0$ and a higher approximation is required ! In view of the linearity of (1), we can expect for $z^\varepsilon := K_\varepsilon^{-1}y^\varepsilon$ (assuming that $v^\varepsilon = K_\varepsilon(v^0 + \varepsilon v^1 + \cdots + v^m)$) an approximation, say z_m^ε, such that $\|z^\varepsilon - z_m^\varepsilon\|_{C([0,T],L^2(0,1))} \leq c_{m,\varepsilon}e^{\frac{(2m+1)\gamma}{2}}$ and therefore an approximation $y_m^\varepsilon := K_\varepsilon z_m^\varepsilon$ of $y^\varepsilon = K_\varepsilon z^\varepsilon$ such that

$$\|y^\varepsilon - y_m^\varepsilon\|_{C([0,T],L^2(0,1))} \leq c_{m,\varepsilon}K_\varepsilon\varepsilon^{\frac{(2m+1)\gamma}{2}} = \mathcal{O}(c_{m,\varepsilon}\varepsilon^{\frac{(2m+1)\gamma}{2}-\frac{3}{2}}).$$

Taking $m \geq 2$ large enough, we can therefore determine a convergent approximation of y^ε provided compatibility conditions between y_0^ε and the control v_k, $0 \leq k \leq m$. The estimation of $c_{m,\varepsilon}$ may however require tedious computations.

A possible alternative to address initial condition like (109) is to preliminary perform a change of variable taking into account the exponential function. The one used in the proof of Proposition 3.1 is prohibited as it makes appears a term with coefficient ε^{-1} (see 104). Still in view of (109), let us assume that the initial condition is of the form $y_0^\varepsilon(x) = c_\varepsilon e^{\frac{M\alpha}{2\varepsilon}}f(x)$ for any function f independent of ε, $\alpha < 0$ and $c_\varepsilon \in \mathbb{R}$. We introduce the following change of variable

$$y^\varepsilon(x,t) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} z^\varepsilon(x,t), \quad l_{\varepsilon,\alpha}(x,t) := \frac{M\alpha}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right). \tag{111}$$

We then check that

$$L_\varepsilon(y^\varepsilon)(x,t) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)}\left(z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + M_\alpha z_x^\varepsilon\right) := c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} L_{\varepsilon,\alpha}(z^\varepsilon)(x,t)$$

with $M_\alpha := M(1-\alpha) > 0$. Consequently, the new variable z^ε solves

$$\begin{cases} L_{\varepsilon,\alpha}(z^\varepsilon) := z_t^\varepsilon - \varepsilon z_{xx}^\varepsilon + M_\alpha z_x^\varepsilon = 0 & \text{in} \quad Q_T, \\ z^\varepsilon(0,\cdot) := \overline{v}^\varepsilon(t) = c_\varepsilon^{-1} e^{-l_{\varepsilon,\alpha}(0,t)} v^\varepsilon, \ z^\varepsilon(L,\cdot) = 0 & \text{on} \quad (0,T), \\ z^\varepsilon(\cdot,0) =: z_0(x) = f(x) & \text{in} \quad (0,L). \end{cases} \tag{112}$$

The initial data is now independent of ε. On the contrary, the control $\overline{v}^\varepsilon$ does a priori on ε. We have thus reported the problem on the control part (which is relevant from a controllability viewpoint).

The asymptotic analysis for z^ε has been done in Section 2: it suffices to replace $M > 0$ by $M_\alpha > 0$. We define the asymptotic approximation of z^ε by

$$z_m^\varepsilon(x,t) = \mathcal{X}_\varepsilon(x) \sum_{k=0}^m \varepsilon^k z^k(x,t) + (1 - \mathcal{X}_\varepsilon(x)) \sum_{k=0}^m \varepsilon^k Z^k\left(\frac{1-x}{\varepsilon}, t\right), \tag{113}$$

where the functions z^k, Z^k are defined as in Section 2. The corresponding control functions are noted by $\overline{v}^k := z^k(0,\cdot)$, $k \geq 0$. Finally, in view of (111), we define the approximation

$$w_m^\varepsilon(x,t) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} z_m^\varepsilon(x,t) \tag{114}$$

so that $L_\varepsilon(w_m^\varepsilon)(x,t) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} L_{\varepsilon,\alpha}(z_m^\varepsilon)(x,t)$. We are then looking for an approximation of the form

$$y^\varepsilon(x,t) = c_\varepsilon e^{\frac{M\alpha}{2\varepsilon}\left(x - \frac{(2-\alpha)Mt}{2}\right)}\left(z^0(x,t) + \varepsilon z^1(x,t) + \varepsilon^2 z^2(x,t) + \cdots\right).$$

The main issue is now to find a set for the control functions $\overline{v}^k$ satisfying the matching condition of (2.6) such that $\|L_\varepsilon y_m^\varepsilon\|_{C([0,T],L^2(0,1))}$ goes to zero with ε. Again, the difficulty is that the control function $\overline{v}^\varepsilon$, through the change of variable (111), may depend on ε. Adapting (55), we write that

$$L_\varepsilon(w_m^\varepsilon)(x,t) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} L_{\varepsilon,\alpha}(z_m^\varepsilon)(x,t) = \sum_{i=1}^5 \underbrace{c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} J_{\varepsilon,\alpha}^i(x,t)}_{:= L_\varepsilon^i(w_m^\varepsilon)(x,t)} \tag{115}$$

with

$$J_{\varepsilon,\alpha}^1(x,t) = -\varepsilon^{m+1} z_{xx}^m(x,t)\mathcal{X}_\varepsilon(x), \quad J_{\varepsilon,\alpha}^2(x,t) = \varepsilon^m (1 - \mathcal{X}_\varepsilon(x)) Z_t^m\left(\frac{1-x}{\varepsilon}, t\right),$$

$$J_{\varepsilon,\alpha}^3(x,t) = M\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{-\gamma}\left(\sum_{k=0}^m \varepsilon^k Z^k\left(\frac{1-x}{\varepsilon}, t\right) - \sum_{k=0}^m \varepsilon^k z^k(x,t)\right),$$

$$J_{\varepsilon,\alpha}^4(x,t) = \mathcal{X}''\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-2\gamma}\left(\sum_{k=0}^m \varepsilon^k Z^k\left(\frac{1-x}{\varepsilon}, t\right) - \sum_{k=0}^m \varepsilon^k z^k(x,t)\right),$$

$$J_{\varepsilon,\alpha}^5(x,t) = 2\mathcal{X}'\left(\frac{1-x}{\varepsilon^\gamma}\right)\varepsilon^{1-\gamma}\left(\varepsilon^{-1}\sum_{k=0}^m \varepsilon^k Z_z^k\left(\frac{1-x}{\varepsilon}, t\right) + \sum_{k=0}^m \varepsilon^k z_x^k(x,t)\right).$$

To go on, let us consider again the simplest case for which $m = 0$. From (10),

$$z^0(x,t) = \begin{cases} z_0(x - M_\alpha t), & x > M_\alpha t, \\ \overline{v}^0\left(t - \dfrac{x}{M_\alpha}\right), & x < M_\alpha t, \end{cases} \tag{116}$$

we get

$$(L^1_\varepsilon(w_0^\varepsilon))(x,t) = -\varepsilon c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} z^0_{xx}(x,t) = -\varepsilon c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)} \begin{cases} z_0^{(2)}(x-M_\alpha t), & x > M_\alpha t, \\ -\dfrac{1}{M_\alpha^2}(\overline{v}^0)^{(2)}\left(t-\dfrac{x}{M_\alpha}\right), & x < M_\alpha t. \end{cases}$$

In view of the identity

$$l_{\varepsilon,\alpha}(x,t) = \frac{M\alpha}{2\varepsilon}\left(-\frac{\alpha x}{2(1-\alpha)} + \frac{(2-\alpha)M}{2}\left(\frac{x}{M_\alpha}-t\right)\right), \tag{117}$$

and that $\alpha<0$, the function $l_{\varepsilon,\alpha}$ is negative on the set $Q^-_{T,\alpha} := \{(x,t)\in Q_T, x > M_\alpha t\} \subset Q^-_{T,0}$. We write

$$\|L^1_\varepsilon(w_0^\varepsilon)\|_{L^2(Q^-_{T,\alpha})} \le \varepsilon c_\varepsilon \|z_0^{(2)}\|_{L^\infty(0,1)} \|e^{l_{\varepsilon,\alpha}(x,t)}\|_{L^2(Q^-_{T,\alpha})}$$

and compute that $\|e^{l_{\varepsilon,\alpha}(x,t)}\|_{L^2(Q^-_{T,\alpha})} = \frac{\sqrt{2}}{\sqrt{M^3|\alpha|^3}}\varepsilon + O(\varepsilon e^{\frac{M\alpha}{2\varepsilon}})$ so that

$$\|L^1_\varepsilon(w_0^\varepsilon)\|_{L^2(Q^-_{T,\alpha})} \le C\varepsilon^2 c_\varepsilon \|z_0^{(2)}\|_{L^\infty(0,1)}.$$

We remark here the benefit of the change of variable (111): with $c_\varepsilon = K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2})$, the norm above goes to zero with ε (in contrast with (110)). Let now $Q^+_{T,\alpha} := \{(x,t)\in Q_T, x < M_\alpha t\}$. We write (using (117)) that

$$\left\|e^{l_{\varepsilon,\alpha}(x,t)}(\overline{v}^0)^{(2)}\left(t-\frac{x}{M_\alpha}\right)\right\|^2_{L^2(Q^+_{T,\alpha})} = \int_0^1 e^{-2\frac{M\alpha^2 x}{4\varepsilon(1-\alpha)}} \int_{\frac{x}{M_\alpha}}^T \left(e^{\frac{\alpha(2-\alpha)M^2}{4\varepsilon}\left(\frac{x}{M_\alpha}-t\right)}(\overline{v}^0)^{(2)}\left(t-\frac{x}{M_\alpha}\right)\right)^2 dt\,dx.$$

The change of variable $\tilde{t} = t - \frac{x}{M_\alpha}$ leads to

$$\begin{aligned}\left\|e^{l_{\varepsilon,\alpha}(x,t)}(\overline{v}^0)^{(2)}\left(t-\frac{x}{M_\alpha}\right)\right\|^2_{L^2(Q^+_{T,\alpha})} &= \int_0^1 e^{-2\frac{M\alpha^2 x}{4\varepsilon(1-\alpha)}} \int_0^{T-\frac{x}{M_\alpha}} \left(e^{l_{\varepsilon,\alpha}(0,\tilde{t})}(\overline{v}^0)^{(2)}(\tilde{t})\right)^2 d\tilde{t}\,dx \\ &\le \|e^{-\frac{M\alpha^2 x}{4\varepsilon(1-\alpha)}}\|^2_{L^2(0,1)} \|e^{l_{\varepsilon,\alpha}(0,t)}(\overline{v}^0)^{(2)}\|^2_{L^2(0,T)} \\ &\le \varepsilon\frac{2(1-\alpha)}{M^2|\alpha|}\|e^{l_{\varepsilon,\alpha}(0,t)}(\overline{v}^0)^{(2)}\|^2_{L^2(0,T)}\end{aligned}$$

leading to $\|L^1_\varepsilon(w_0^\varepsilon)\|_{L^2(Q^+_{T,\alpha})} \le Cc_\varepsilon\varepsilon^{3/2}\|e^{l_{\varepsilon,\alpha}(0,t)}(\overline{v}^0)^{(2)}\|_{L^2(0,T)}$ for some $C>0$ and finally to

$$\|L^1_\varepsilon(w_0^\varepsilon)\|_{L^2(Q_T)} \le Cc_\varepsilon\varepsilon^{3/2}\left(\varepsilon^{1/2}\|z_0^{(2)}\|_{L^\infty(0,1)} + \|e^{l_{\varepsilon,\alpha}(0,t)}(\overline{v}^0_\varepsilon)^{(2)}\|_{L^2(0,T)}\right). \tag{118}$$

Let us now consider the second term $J^2_{\varepsilon,\alpha}$ in the expansion 115. Adapting (15), we have $Z^0(z,t) = z^0(1,t)(1-e^{-M_\alpha z})$. Therefore

$$\begin{aligned}\|e^{l_{\varepsilon,\alpha}}J^2_{\varepsilon,\alpha}\|^2_{L^2(Q_T)} &= \int_0^1\int_0^T e^{2l_{\varepsilon,\alpha}(x,t)}(z^0_t(1,t))^2(1-\mathcal{X}_\varepsilon)(1-e^{-M_\alpha z})dtdx \\ &\le \int_{1-2\varepsilon^\gamma}^1\int_0^T e^{2l_{\varepsilon,\alpha}(x,t)}(z^0_t(1,t))^2 dtdx \\ &= M_\alpha^2\int_{1-2\varepsilon^\gamma}^1\int_0^{\frac{1}{M_\alpha}} e^{2l_{\varepsilon,\alpha}(x,t)}\Big((z_0^{(1)}(1-M_\alpha t)\Big)^2 dtdx \\ &+ \int_{1-2\varepsilon^\gamma}^1\int_{\frac{1}{M_\alpha}}^T e^{2l_{\varepsilon,\alpha}(x,t)}\Big((\overline{v}_0^{(1)}(t-\frac{1}{M_\alpha})\Big)^2 dtdx.\end{aligned}$$

We check that $l_{\varepsilon,\alpha}(x,t) < -\frac{M\alpha}{2\varepsilon}\bigg(\frac{\alpha}{2(1-\alpha)} + 2\varepsilon^\gamma\bigg) < 0$ for all $(x,t) \in (1-2\varepsilon^\gamma, 1) \times (0, 1/M_\alpha)$ so that the first term is negligible. For the second term, we make the change of variable $\tilde{t} = t - 1/M_\alpha$; we then check that

$$l_{\varepsilon,\alpha}(x,t) = \frac{M\alpha}{2\varepsilon}\bigg(\frac{(\alpha-2)}{2(1-\alpha)} + x\bigg) + e^{l_{\varepsilon,\alpha}(0,\tilde{t})} := g_\varepsilon(x) + e^{l_{\varepsilon,\alpha}(0,\tilde{t})}$$

and then write

$$\begin{aligned}\int_{1-2\varepsilon^\gamma}^1 \int_{\frac{1}{M_\alpha}}^T e^{2l_{\varepsilon,\alpha}(x,t)}\bigg((\overline{v}_0^{(1)}(t-\frac{1}{M_\alpha})\bigg)^2 dtdx &= \int_{1-2\varepsilon^\gamma}^1 e^{2g_\varepsilon(x)}dx \int_0^{T-\frac{1}{M_\alpha}} e^{-2\frac{\alpha(2-\alpha)}{4\varepsilon}M^2 t}(\overline{v}_0^{(1)}(t))^2 dtdx \\ &\le \|e^{g_\varepsilon(x)}\|^2_{L^2(1-2\varepsilon^\gamma,1)} \|e^{l_{\varepsilon,\alpha}(0,t)}\overline{v}_0^{(1)}\|^2_{L^2(0,T)} \\ &\le 2\varepsilon^\gamma e^{\frac{M\alpha}{\varepsilon}\big(-\frac{\alpha}{2(1-\alpha)}-2\varepsilon^\gamma\big)} \|e^{l_{\varepsilon,\alpha}(0,t)}\overline{v}_0^{(1)}\|^2_{L^2(0,T)}\end{aligned}$$

since $g_\varepsilon(x) \le \frac{M\alpha}{2\varepsilon}\big(-\frac{\alpha}{2(1-\alpha)} - 2\varepsilon^\gamma\big)$ for all $x \in (1-2\varepsilon^\gamma, 1)$. Moreover, the bound $\frac{M\alpha}{2\varepsilon}\big(-\frac{\alpha}{2(1-\alpha)} - 2\varepsilon^\gamma\big)$ is strictly negative (for ε small enough), so that this term is once again negligible. With similar arguments, we conclude that the terms $\|e^{l_{\varepsilon,\alpha}} J^i_{\varepsilon,\alpha}\|_{L^2(Q_T)}$, $i = 3, 4$ and $i = 5$ are exponentially small with respect to ε.

Therefore, we have the following result :

Theorem 4.1 *Let $\alpha < 0$, $c_\varepsilon \in \mathbb{R}$, let $l_{\varepsilon,\alpha}(x,t) := \frac{M\alpha}{2\varepsilon}\bigg(x - \frac{(2-\alpha)Mt}{2}\bigg)$, $f \in C^1([0,1])$ and $v^\varepsilon \in C^1([0,T])$. Let us consider the problem*

$$\begin{cases} y_t^\varepsilon - \varepsilon y_{xx}^\varepsilon + M y_x^\varepsilon = 0, & (x,t) \in Q_T, \\ y^\varepsilon(0,t) = v^\varepsilon(t), & t \in (0,T), \\ y^\varepsilon(1,t) = 0, & t \in (0,T), \\ y^\varepsilon(x,0) = y_0^\varepsilon(x) := c_\varepsilon e^{l_{\varepsilon,\alpha}(x,0)} f(x), & x \in (0,1), \end{cases} \tag{119}$$

and assume that y_0^ε and v^ε satisfies the compatibility conditions

$$y_0^\varepsilon(0) = v^\varepsilon(0), \qquad \frac{M^2\alpha^2}{4\varepsilon} v^\varepsilon(0) + (v^\varepsilon)^{(1)}(0) = -M(1-\alpha)(y_0^\varepsilon)^{(1)}(0). \tag{120}$$

Let then w_0^ε be defined as follows :

$$w_0^\varepsilon(x,t) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,t)}\bigg(\mathcal{X}_\varepsilon(x) z^0(x,t) + (1-\mathcal{X}_\varepsilon(x)) Z^0(x,t)\bigg)$$

where z^0 is given by (116) associated to the initial condition $z_0(x) = f(x)$ and control $\overline{v}^0 = c_\varepsilon^{-1} e^{-l_{\varepsilon,\alpha}(0,t)} v^\varepsilon(t)$ and where $Z^0(x,t) = z^0(1,t)(1-e^{-M_\alpha z})$.

Then $w_0^\varepsilon \in C^1(\overline{Q_T})$ and there exists two constants $C, c > 0$ independent of ε such that

$$\|y^\varepsilon - w_0^\varepsilon\|_{C([0,T],L^2(0,1))} \le C c_\varepsilon \varepsilon^{3/2}\bigg(\varepsilon^{1/2}\|f^{(2)}\|_{L^\infty(0,1)} + \|e^{l_{\varepsilon,\alpha}(0,t)}(\overline{v}^0)^{(2)}\|_{L^2(0,T)}\bigg) + c_\varepsilon \mathcal{O}(e^{-\frac{c}{\varepsilon}}). \tag{121}$$

Proof- Conditions (120) imply the property $f(0) = \overline{v}(0)$ and $(\overline{v})'(0) = -M_\alpha f'(0)$. Therefore, in view of Lemma 2.6 for $m = 0$, z^0 and Z^0 and then w_0^ε belongs to $C^1(\overline{Q_T})$. Moreover, the function $z^\varepsilon := y^\varepsilon - w_0^\varepsilon$ satisfies the boundary value problem :

$$\begin{cases} L_\varepsilon(z^\varepsilon) = -L_\varepsilon(w_0^\varepsilon), & (x,t) \in Q_T, \\ z^\varepsilon(0,t) = z^\varepsilon(1,t) = 0, & t \in (0,T), \\ z^\varepsilon(x,0) = c_\varepsilon e^{l_{\varepsilon,\alpha}(x,0)}\bigg(f(x) - Z^0(x,0)\bigg)(1-\mathcal{X}_\varepsilon), & x \in (0,1). \end{cases}$$

and therefore $\|y^\varepsilon - w_0^\varepsilon\|_{C([0,T],L^2(0,1))} \le C\Big(\|L_\varepsilon(w_0^\varepsilon)\|_{L^2(Q_T)} + \|z^\varepsilon(x,0)\|_{L^2(0,1)}\Big)$. The L^2-norm of $z^\varepsilon(\cdot,0)$ is exponentially small, since $\|z^\varepsilon(\cdot,0)\|_{L^2(0,1)} = \|z^\varepsilon(\cdot,0)\|_{L^2(1-2\varepsilon^\gamma,1)} \le e^{\frac{M\alpha(1-2\varepsilon^\gamma)}{2\varepsilon}}\|f - Z^0(\cdot,0)\|_{L^2(0,1)}$. (121) then follows from (118). □

For the initial condition (109) for which $c_\varepsilon = K_\varepsilon = \mathcal{O}(\varepsilon^{-3/2})$, $f(x) = \sin(\pi x)$, $\alpha = -1$, (121) writes

$$\|y^\varepsilon - w_0^\varepsilon\|_{C([0,T],L^2(0,1))} \le C\Big(\varepsilon^{1/2}\|\pi^2\sin(\pi x)\|_{L^\infty(0,1)} + \|e^{\frac{3M^2 t}{4\varepsilon}}(\overline{v}^0)^{(2)}\|_{L^2(0,T)}\Big) + c_\varepsilon\mathcal{O}(e^{-\frac{c}{\varepsilon}})$$

where $\overline{v}^0$ is $C^1([0,T])$ function such that $\overline{v}^0(0) = 0$, $(\overline{v}^0)^{(1)}(0) = -2\pi$. It suffices then that $\|e^{\frac{3M^2 t}{4\varepsilon}}(\overline{v}^0)^{(2)}\|_{L^2(0,T)}$ goes to zero with ε to ensure the approximation.

Actually, estimate (121) is mainly interesting from a controllability viewpoint as we may choose $\overline{v}^0$ such that non only $\|e^{l_{\varepsilon,\alpha}(0,t)}(\overline{v}^0)^{(2)}\|_{L^2(0,T)}$ but also $\|w_0^\varepsilon(\cdot,T)\|_{L^2(0,T)}$ goes to zero with ε. For instance, if $\|w_0^\varepsilon(\cdot,T)\|_{L^2(0,T)}$ vanishes then $\|y^\varepsilon(\cdot,T)\|_{L^2(0,1)} \le C\|y^\varepsilon - w_0^\varepsilon\|_{C([0,T],L^2(0,1))}$ and v^ε is an approximate control at time T for y^ε solution of (119) with initial data $c_\varepsilon e^{\frac{M\alpha x}{2\varepsilon}}f(x)$. In view of (114), $w_0^\varepsilon(\cdot,T) = 0$ if and only if $z_0^\varepsilon(\cdot,T) = 0$. $z_0^\varepsilon = z^0$ given by (116) is solution of a transport equation and vanishes at time T if and only the support of the control function $\overline{v}^0$ is in $[0, T - \frac{1}{M_\alpha}]$.

Let $\eta > 0$ and $\beta \in]0, T - \frac{1}{M_\alpha}]$. We choose the control function $\overline{v}^0$ as the unique solution of the following ordinary differential equation

$$\begin{cases} (\overline{v}^0)^{(2)}(t) = (C_1^\varepsilon + C_2^\varepsilon t)e^{\frac{-\eta}{4\varepsilon}t}e^{-l_{\varepsilon,\alpha}(0,t)}, \quad t \in [0,\beta], \\ \overline{v}^0(0) = z_0^\varepsilon(0), \quad \overline{v}^0(\beta) = 0, \\ (\overline{v}^0)^{(1)}(0)) = -M_\alpha(z_0^\varepsilon)'(0), \quad (\overline{v}^0)^{(1)}(\beta) = 0, \end{cases} \tag{122}$$

for some constants C_1^ε and C_2^ε. Problem (122) admits a unique solution given by

$$\overline{v}^0(t) = \frac{kC_1^\varepsilon - 2C_2^\varepsilon + kC_2^\varepsilon t}{k^3}e^{kt} + C_3^\varepsilon t + C_4^\varepsilon, \quad k := \frac{-\eta + \alpha(2-\alpha)M^2}{4\varepsilon} \tag{123}$$

with

$$\begin{cases} C_1^\varepsilon := -\overline{v}^0(0)\dfrac{k^2\left(-\mathrm{e}^{k\beta} + \mathrm{e}^{k\beta}k\beta + 1\right)}{\left(\mathrm{e}^{k\beta}\right)^2 - 2\,\mathrm{e}^{k\beta} + 1 - \beta^2k^2\mathrm{e}^{k\beta}} - (\overline{v}^0)^{(1)}(0))\dfrac{k\left(-2\,\mathrm{e}^{k\beta}k\beta + 2\,\mathrm{e}^{k\beta} - 2 + \beta^2k^2\mathrm{e}^{k\beta}\right)}{\left(\mathrm{e}^{k\beta}\right)^2 - 2\,\mathrm{e}^{k\beta} + 1 - \beta^2k^2\mathrm{e}^{k\beta}}, \\ C_2^\varepsilon := \overline{v}^0(0)\dfrac{k^3\left(\mathrm{e}^{k\beta} - 1\right)}{\left(\mathrm{e}^{k\beta}\right)^2 - 2\,\mathrm{e}^{k\beta} + 1 - \beta^2k^2\mathrm{e}^{k\beta}} + (\overline{v}^0)^{(1)}(0))\dfrac{k^2\left(-\mathrm{e}^{k\beta} + \mathrm{e}^{k\beta}k\beta + 1\right)}{\left(\mathrm{e}^{k\beta}\right)^2 - 2\,\mathrm{e}^{k\beta} + 1 - \beta^2k^2\mathrm{e}^{k\beta}}, \\ C_3^\varepsilon := \overline{v}^0(0)\dfrac{\mathrm{e}^{k\beta}k^2\,\beta}{\left(\mathrm{e}^{k\beta}\right)^2 - 2\,\mathrm{e}^{k\beta} + 1 - \beta^2k^2\mathrm{e}^{k\beta}} + (\overline{v}^0)^{(1)}(0))\dfrac{\mathrm{e}^{k\beta}\left(\mathrm{e}^{k\beta} - k\beta - 1\right)}{\left(\mathrm{e}^{k\beta}\right)^2 - 2\,\mathrm{e}^{k\beta} + 1 - \beta^2k^2\mathrm{e}^{k\beta}}, \\ C_4^\varepsilon := \overline{v}^0(0)\dfrac{\mathrm{e}^{k\beta}\left(k\beta - 1 + \mathrm{e}^{k\beta} - \beta^2k^2\right)}{\left(\mathrm{e}^{k\beta}\right)^2 - 2\,\mathrm{e}^{k\beta} + 1 - \beta^2k^2\mathrm{e}^{k\beta}} + (\overline{v}^0)^{(1)}(0))\dfrac{\mathrm{e}^{k\beta}k\beta^2}{\left(\mathrm{e}^{k\beta}\right)^2 - 2\,\mathrm{e}^{k\beta} + 1 - \beta^2k^2\mathrm{e}^{k\beta}}. \end{cases} \tag{124}$$

With this choice, we have $\|e^{l_{\varepsilon,\alpha}(0,t)}(\overline{v}^0)^{(2)}\|_{L^2(0,T)} = \|(C_1^\varepsilon + C_2^\varepsilon t)e^{-\frac{\eta}{4\varepsilon}t}\|_{L^2(0,\beta)}$. Now, from (124), we obtain, assuming that the constant β is independent of ε, that

$$\begin{cases} C_1^\varepsilon \approx -\overline{v}^0(0)k^2 + 2k(\overline{v}^0)^{(1)}(0) \approx -\overline{v}^0(0)\varepsilon^{-2} + 2\varepsilon^{-1}(\overline{v}^0)^{(1)}(0), \\ C_2^\varepsilon \approx -\overline{v}^0(0)k^3 + k^2(\overline{v}^0)^{(1)}(0) \approx \overline{v}^0(0)\varepsilon^{-3} + \varepsilon^{-2}(\overline{v}^0)^{(1)}(0), \end{cases} \tag{125}$$

which implies, after some computations, the estimate $\|e^{l_{\varepsilon,\alpha}(0,t)}(\overline{v}^0)^{(2)}\|_{L^2(0,T)} \approx \varepsilon^{-3/2}|\overline{v}^0(0)| + \varepsilon^{-1/2}|(\overline{v}^0)^{(1)}(0)|$. Finally, for such control function $\overline{v}^0$, (121) leads to

$$\|y^\varepsilon - w_0^\varepsilon\|_{C([0,T],L^2(0,1))} \le Cc_\varepsilon\varepsilon^{3/2}\Big(\varepsilon^{1/2}\|f^{(2)}\|_{L^\infty(0,1)} + \varepsilon^{-3/2}|\overline{v}^0(0)| + \varepsilon^{-1/2}|(\overline{v}^0)^{(1)}(0)|\Big) + c_\varepsilon\mathcal{O}(e^{-\frac{c}{\varepsilon}}).$$

Consequently, in the particular case for which $|\overline{v}^0(0)| = 0$ and $c_\varepsilon \varepsilon \to 0$ as $\varepsilon \to 0$, $\overline{v}^0$ is an approximate control for y^ε.

As an illustration, figures 4 plots the control $v^0(t) = e^{l_{\varepsilon,\alpha}(0,t)}\overline{v}^0(t)$ associated to the initial condition $y_0^\varepsilon = e^{-\frac{Mx}{2\varepsilon}}\sin(\pi x)$ for $T = 1$, $\eta = 1$, $\beta := \frac{1}{M_\alpha} = \frac{1}{2}$ and $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$. By construction, the control function gets concentrated at $x = 0$ as $\varepsilon \to 0$.

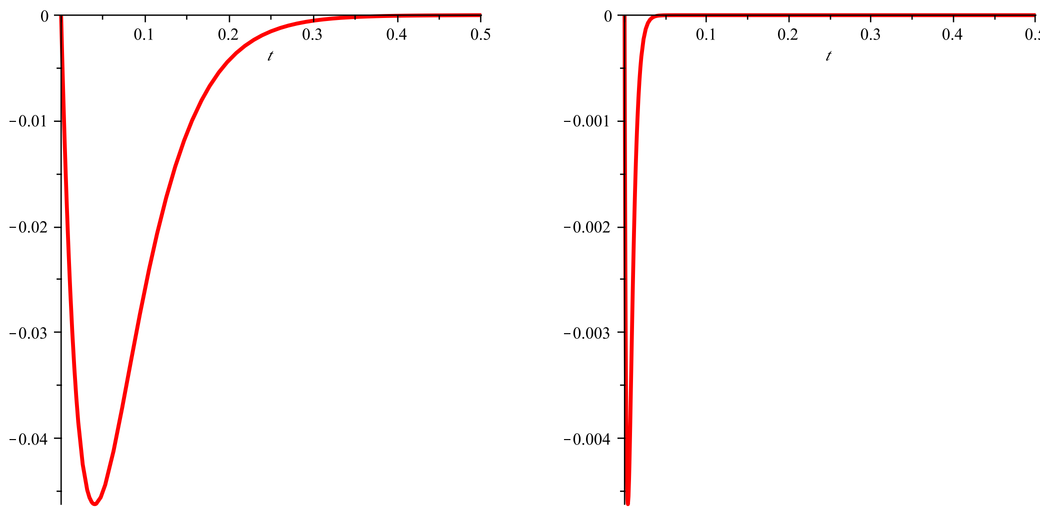

Figure 3: Control $v^0(t)$ for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$ associated to $y_0^\varepsilon(x) = e^{\frac{-Mx}{2\varepsilon}}\sin(\pi x)$.

Remark 6 *We check that the control function v^0 (and a fortiori $\overline{v}^0$) is uniformly bounded in $L^\infty(0,T)$ with respect to ε; we have*

$$v^0(t) = \frac{kC_1^\varepsilon - 2C_2^\varepsilon + kC_2^\varepsilon t}{k^3}e^{-\frac{\eta t}{4\varepsilon}} + (C_3^\varepsilon t + C_4^\varepsilon)e^{-\frac{\gamma M^2 t}{4\varepsilon}}, \quad t \in [0,\beta].$$

The first term is bounded in view of 125), the term $-\eta t/(4\varepsilon)$ being negative. The second term behaves for ε small as follows :

$$(C_3^\varepsilon t + C_4^\varepsilon)e^{-\frac{\gamma M^2 t}{4\varepsilon}} \approx k\beta\Big(\big(\overline{v}^0(0)k - (\overline{v}^0)^{(1)}(0)\big)t + \beta\big(-\beta\overline{v}^0(0) + (\overline{v}^0)^{(1)}(0)\big)\Big)e^{-\frac{\eta\beta}{4\varepsilon}}e^{\frac{\gamma M^2}{4\varepsilon}(\beta - t)}$$

using from (124) that

$$C_3^\varepsilon \approx k\beta e^{k\beta}\big(k\overline{v}^0(0) - (\overline{v}^0)^{(1)}(0)\big), \quad C_4^\varepsilon \approx k\beta^2 e^{k\beta}\big(-\beta\overline{v}^0(0) + (\overline{v}^0)^{(1)}(0)\big).$$

Moreover, by construction (see (122), we also get that v^0 and $\overline{v}^0$ are uniformly bounded in $C^1([0,T])$. $(\overline{v}^0(t))^{(1)}(t) = C_3^\varepsilon + k^{-2}(kC_1^\varepsilon - C_2^\varepsilon + kC_2^\varepsilon t)e^{kt}$.

5 About the control of minimal L^2-norm - Conclusion and Perspective

We have derived an asymptotic expansion at any order m of the solution y^ε of an advection-diffusion equation with respect to the diffusion parameter ε. The matching asymptotic method allows to describe the boundary layer of the solution at the right extremity of the interval. As is usual, the asymptotic analysis requires the initial and boundary conditions to be regular enough. This is not restrictive as y^ε solves a parabolic type equation. In an essential way, we have also assumed compatibility equations between these conditions at the point $(x,t)=(0,0)$ where the main characteristic of equation $Lx-Mt=0$ start. This allows to get rid off the boundary layer for y^ε on this characteristic. This also allows to obtain a regular approximation w_m^ε of y^ε so that the norm $\|y^\varepsilon-w_m^\varepsilon\|_{C([0,T],L^2(0,L))}$ is of size $\mathcal{O}(\varepsilon^m)$. As expected, the approximation w_m^ε is mainly the sum of $m+1$ explicit solutions of transport equations. As a matter of fact, the diffusion property of y^ε which is so essential in the controllability property, is lost in w_m^ε. Nevertheless, the approximation w_m^ε is useful to construct explicit and regular approximate null controls for y^ε as soon as the controllability time satisfies $T\geq L/M$.

The next step is to use such asymptotic analysis in the optimality system (6) which characterizes the unique control of minimal $L^2(0,T)$-norm, T,ε and the initial condition y_0 (assumed independent of ε) being fixed. Let us focus on the optimality equation $v^\varepsilon(t)=\varepsilon\varphi_x^\varepsilon(0,t)$ which links the forward and backward equation. Using the inner expansion for φ^ε (see Section 2.5), this equality rewrites as follow

$$v^0(t)+\varepsilon\, v^1(t)+\cdots=\Phi_z^0(0,t)+\varepsilon\,\Phi_z^1(0,t)+\cdots,\quad \forall t\in(0,T).$$

At the zero order, we get therefore the equality $v^0(t)=\Phi_z^0(0,t)$ leading, using (94) and (97) simultaneously, to

$$v^0(t)=M\varphi^0(0,t)=\begin{cases}M\varphi_T^0(M(T-t)), & t\in]T-1/M,T],\\ 0, & t\in[0,T-1/M].\end{cases}\tag{126}$$

The function φ^0 defined in Q_T is given by (94). It $T>1/M$, the last equality contradicts the matching conditions (46), notably $v^0(0)=y_0(0)$, unless that $y_0(0)=0$! If $T=1/M$, we have $v^0(t)=M\varphi_T^0(1-Mt)$, $t\in[0,1/M]$ and in particular $v^0(0)=\varphi_T^0(1)$. But again, this contradicts (101) unless $v^0(0)=0$ (and so $y_0(0)=0$). Assuming $y_0(0)=0$, we may determine the optimal function φ_T^0 by developing the conjugate functional $J_\varepsilon^\star$ given by

$$J_\varepsilon^\star(\varphi_T^\varepsilon):=\frac{1}{2}\int_0^T(\varepsilon\varphi_x^\varepsilon(0,t))^2dt-(y_0,\varphi^\varepsilon(\cdot,0))_{H^{-1}(0,1),H_0^1(0,1)}.$$

We easily obtain $J_\varepsilon^\star(\varphi_T^\varepsilon)=J_0^\star(\varphi_T^0)+\varepsilon\ldots$ with

$$J_0^\star(\varphi_T^0):=\frac{1}{2}\|v^0\|^2_{L^2(0,T)}-\bigg(y_0,\mathcal{X}_\varepsilon\varphi^0(x,0)+(1-\mathcal{X}_\varepsilon)\Phi^0(x,0)\bigg)_{L^2(0,1)}$$

which is simply $J_0^\star(\varphi_T^0)=\|v^0\|^2_{L^2(0,T)}/2$ since $\varphi^0(\cdot,0)=0$ (see (94)) and $\Phi^0(\cdot,0)=0$ (see (97)). The minimization of $J_\varepsilon^\star$ at the first order, that is the minimization of $J_0^\star$ leads to $\varphi_T^0=0$, i.e. $v^0\equiv 0$. Remark that since φ^0 solves a transport equation which separates the space-time domain Q_T into two parts $\{(x,t)\in Q_T, x-Mt>0\}$ and $\{(x,t)\in Q_T, x-Mt<0\}$ and since φ^0 vanishes at $x=1$, the first order control term v^0 does not "see" the initial condition y_0. Repeating the arguments and assuming that $(y_0)^{(m)}(0)=0$, we obtain that $v^m\equiv 0$ on $(0,T)$ for all $m\geq 1$. We conclude that, for $T\geq 1/M$, the norm of the control of minimal $L^2(0,T)$-norm associated to initial conditions y_0 satisfying $y_0^{(m)}(0)=0$ for all $m\geq 0$ and (68) vanishes as ε goes to 0.

If we do not assume $y_0(0)=0$, then (126) leads to incompatibility and our asymptotic analysis is not effective to address the optimality system (6). To avoid this difficulty, we must relax the matching conditions (47) and (101) and therefore take into account the second boundary layer occurring for y^ε and φ^ε on the characteristic lines $\{(x,t)\in Q_T, Lx-Mt=0\}$ and $\{(x,t)\in Q_T, Lx-M(t-T)-1=0\}$ respectively. This will be done in a forthcoming work. The negative case $M<0$, which is similar, will be addressed as well.

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