

Resolution of implicit time schemes for the Navier-Stokes system through a least-squares method.

Jérôme Lemoine, Arnaud Munch

▶ **To cite this version:**

Jérôme Lemoine, Arnaud Munch. Resolution of implicit time schemes for the Navier-Stokes system through a least-squares method.. 2019. hal-01996429v2

HAL Id: hal-01996429

https://uca.hal.science/hal-01996429v2

Preprint submitted on 13 Feb 2019 (v2), last revised 29 Dec 2020 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RESOLUTION OF IMPLICIT TIME SCHEMES FOR THE NAVIER-STOKES SYSTEM THROUGH A LEAST-SQUARES METHOD

JÉRÔME LEMOINE AND ARNAUD MÜNCH

Abstract. Implicit time schemes reduce the numerical resolution of the Navier-Stokes system to multiple resolutions of steady Navier-Stokes equations. We analyze a least-squares method, introduced by Glowinski in 1979, to solve the steady Navier-Stokes equation. Precisely, we show that any minimizing sequences (constructed by gradient type methods) for a least-squares functional converges strongly toward solutions, assuming the initial guess in an explicit ball dependent of the time step and of the viscosity constant. The resulting method is faster and more robust than the Newton method used to solve the weak variational formulation for the Navier-Stokes system. Numerical experiments support our analysis.

Key Words. Steady Navier-Stokes system, Least-squares approach, Gradient method.

AMS subject classifications. 35Q30, 93E24.

1. Introduction - Motivation

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or $d = 3$ be a bounded connected open set whose boundary $\partial\Omega$ is Lipschitz. We denote by $\boldsymbol{V} = \{v \in \mathcal{D}(\Omega)^d, \nabla \cdot v = 0\}$, $\boldsymbol{H}$ the closure of $\boldsymbol{V}$ in $L^2(\Omega)^d$ and $\boldsymbol{\mathcal{V}}$ the closure of $\boldsymbol{V}$ in $H^1(\Omega)^d$. Let $T > 0$.

The Navier-Stokes system describes a viscous incompressible fluid flow in the bounded domain Ω during the time interval $(0, T)$ submitted to the external force F. It reads as follows :

$$\begin{cases} u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = F, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0, T), \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0, \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where u is the velocity of the fluid, p its pressure and ν is the viscosity constant assumed smaller than one. We refer to [20]. This work is concerned with the numerical approximation of (1.1) through time marching implicit schemes of the form

$$\begin{cases} y^0(\cdot, 0) = u_0 \quad \text{in} \quad \Omega, \\ \dfrac{y_\theta^{n+1} - y^n}{\theta\delta t} - \nu\Delta y_\theta^{n+1} + (y_\theta^{n+1} \cdot \nabla)y_\theta^{n+1} + \nabla\pi_\theta^{n+1} = \dfrac{1}{\delta t}\displaystyle\int_{t_n}^{t_{n+1}} F(\cdot, s)ds, \quad n \geq 0, \\ \nabla \cdot y_\theta^{n+1} = 0 \quad \text{in} \quad \Omega, \quad n \geq 0, \\ y_\theta^{n+1} = 0 \quad \text{on} \quad \partial\Omega, \quad n \geq 0, \\ y^{n+1} = \theta^{-1}(y_\theta^{n+1} - (1 - \theta)y^n), \quad n \geq 0 \end{cases} \tag{1.2}$$

where θ is a parameter in $(0, 1]$ and where $\{t_n\}_{n=0\ldots N}$, for a given $N \in \mathbb{N}$, is a uniform discretization of the time interval $(0, T)$. $\delta t = T/N$ is the time discretization step. The case $\theta = 1$, for which $y^n = y_\theta^n$ for all n, corresponds to the backward Euler scheme studied for instance in

Date: February, 1 2019.

Laboratoire de Mathématiques, Université Clermont Auvergne, UMR CNRS 6620, Campus des Cézeaux, 63177 Aubière, France. e-mail: jerome.lemoine@uca.fr.

Laboratoire de Mathématiques, Université Clermont Auvergne, UMR CNRS 6620, Campus des Cézeaux, 63177 Aubière, France. e-mail: arnaud.munch@uca.fr.

[20, chapter 3, section 4]. It is proved there that the piecewise linear interpolation (in time) of $\{y^n\}_{n\in[0,N]}$ weakly converges in $L^2(0,T,\boldsymbol{V})$ toward a solution u of (1.1) as δt goes to zero. It achieves a first order convergence with respect to δt. We also refer to [21] for a stability analysis of the scheme in long time. The case $\theta = 1/2$ corresponds to a Crank-Nicolson scheme and allows to achieve a second order convergence. We refer to [18] and the references therein.

The determination of y^{n+1} from y^n requires the resolution of a nonlinear partial differential equation. Precisely, y_θ^{n+1} together with the pressure π_θ^{n+1}, solve the following problem: find $y \in \boldsymbol{V}$ and $\pi \in L^2_0(\Omega)$, solution of

$$\begin{cases} \alpha\, y - \nu\Delta y + (y\cdot\nabla)y + \nabla\pi = f + \alpha\, g, \quad \nabla\cdot y = 0 \quad \text{in } \Omega, \\ y = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with

$$\alpha = \frac{1}{\theta\delta t} > 0, \quad f = \frac{1}{\delta t}\int_{t_n}^{t_{n+1}} F(\cdot,s)ds, \quad g = y^n. \tag{1.4}$$

Recall that for any $f \in H^{-1}(\Omega)^d$ and $g \in L^2(\Omega)^d$, there exists at least one $(y,\pi) \in \boldsymbol{V}\times L^2_0(\Omega)$ solution of (1.3). $L^2_0(\Omega)$ stands for the space of functions in $L^2(\Omega)$ with zero means. Moreover, if $\|g\|_2^2 + \alpha^{-1}\nu^{-1}\|f\|^2_{H^{-1}(\Omega)^d}$ is small enough, then the couple (y,π) is unique (see Proposition 2.2 for a more precise statement). Here and in the sequel, $\|\cdot\|_2$ stands for the L^2 norm $\|\cdot\|_{L^2(\Omega)^d}$.

The approximation of solutions of (1.3) can be performed using Newton's type methods (see for instance [17, Section 10.3]) for the weak formulation of (1.3); this consists in solving iteratively the following variational problem : find $y \in \boldsymbol{V}$ solution of

$$\begin{aligned} F(y,z) := \int_\Omega \alpha\, y\cdot z + \nu\nabla y\cdot\nabla z + (y\nabla)y\cdot z \\ - < f, z >_{H^{-1}(\Omega)^d\times H^1_0(\Omega)^d} - \alpha\int_\Omega g\cdot z = 0, \quad \forall z \in \boldsymbol{V}. \end{aligned} \tag{1.5}$$

Alternatively, we may also employ least-squares methods which consists in minimizing a quadratic functional, which measure how an element y is close to the solution. Precisely, we define the functional $E : \boldsymbol{V} \to \mathbb{R}^+$ by putting

$$E(y) = \frac{1}{2}\int_\Omega \alpha|v|^2 + |\nabla v|^2 \tag{1.6}$$

where the corrector v, together with the pressure, is the unique solution in $\boldsymbol{V}\times L^2_0(\Omega)$ of the linear boundary value problem:

$$\begin{cases} \alpha v - \Delta v + \nabla\pi + (\alpha y - \nu\Delta y + (y\cdot\nabla)y - f - \alpha g) = 0, \quad \nabla\cdot v = 0 \quad \text{in} \quad \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.7}$$

Least-squares methods to solve nonlinear boundary value problems have been the subject of intensive developments in the last decades, as they present several advantages, notably on computational and stability viewpoints. We refer to the books [1, 7]. The minimization of the functional E over $\boldsymbol{V}$ leads to a so-called H^{-1}-least squares method. This method has been introduced and numerically implemented in [2] to approximate the solutions of (1.1) through the scheme (1.2) with $\theta = 1$. However, there is no analysis nor mathematical justification of the method in [2]. Let us mention [4, Chapter 4, Section 6] which studied later the use of a least-squares strategy to solve a steady Navier-Stokes equation without incompressibility constraint. The main reason of the present work is therefore to show that minimizing sequences for the so-called error functional E do actually converge strongly to a solution of (1.3). This guaranties the determination of y^{n+1} from y^n, solution of (1.2) and therefore an approximation of the solution of the unsteady Navier-Stokes equation (1.1).

The paper is organized as follows. In Section 2, we analyze a least-squares method for weak solutions y of (1.3), under some conditions on g, f, α and ν. We show in particular the strong

convergence of any minimizing sequence for E starting closed enough to a solution. The analysis notably exhibits a descent direction Y_1 (defined in (2.11)) for which the derivative of E is colinear to E (see eq. 2.18). In Section 3, we apply the least-squares approach to compute recursively the sequence $\{y^n\}_{n\in\mathbb{N}}$ defined by (1.2) and notably show that two consecutive elements of the sequence are closed for the $L^2(\Omega)^d$-norm as soon as δt is small enough. This suggests, in order to calculate the element y^{n+1} from y^n, to initialize the minimizing sequence of the least-squares functional with the element y^n (as done in [2]). In Section 4, we derive the conjugate gradient algorithm commonly used for this kind of functional. The minimizing sequence is then construct from the gradient of the functional with respect to the $\boldsymbol{V}$-norm. We also consider minimizing sequences defined from the element Y_1 mentioned above and remark that the corresponding algorithm leads to a generalization of the Newton algorithm, when used to solve directly the weak variational formulation (1.5) associated to (1.3). Numerical experiments in Section 5 confirms the efficiency of the method based on the element Y_1, in particular for very small values of the viscosity constant ν.

2. Analysis of a Least-squares method for a steady Navier-Stokes equation

We analyse in this section a least-squares method to solve the steady Navier-Stokes equation 1.3: we follow and improve [11] where the particular case $\alpha = 0$ is addressed.

In the following, we repeatedly use the following classical estimates.

Lemma 2.1. *Let any $u, v \in \boldsymbol{V}$. If $d = 2$, then there exists a constant $c = c(\Omega)$ such that*

$$\int_\Omega u \cdot \nabla v \cdot u \leq c\|u\|_2\|\nabla v\|_2\|\nabla u\|_2. \tag{2.1}$$

If $d = 3$, then

$$\int_\Omega u \cdot \nabla v \cdot u \leq c\|u\|_2^{1/2}\|\nabla v\|_2\|\nabla u\|_2^{3/2}. \tag{2.2}$$

Proof. For the estimate (2.2), see [20]. If, $d = 2$ and if $u, v, w \in \boldsymbol{V}$, denoting $\tilde{u}, \tilde{v}$ and $\tilde{w}$ their extension to 0 in $\mathbb{R}^2$, we have, see [3] and [19]

$$\begin{aligned}|\int_\Omega u.\nabla v.w| = |\int_\Omega \tilde{u}.\nabla\tilde{v}.\tilde{w}| &\leq \|\tilde{u}.\nabla\tilde{v}\|_{\mathcal{H}^1(\mathbb{R}^2)}\|\tilde{w}\|_{BMO(\mathbb{R}^2)} \leq c\|\tilde{u}\|_2\|\nabla\tilde{v}\|_2\|\tilde{w}\|_{H^1(\mathbb{R}^2)}\\ &\leq c\|u\|_2\|\nabla v\|_2\|w\|_{H^1(\Omega)^2} \leq c\|u\|_2\|\nabla v\|_2\|\nabla w\|_2.\end{aligned}$$

Taking $w = u$ gives (2.1).

□

Let $f \in H^{-1}(\Omega)^d$, $g \in L^2(\Omega)^d$ and $\alpha \in \mathbb{R}_+^\star$. The weak formulation of (1.3) reads as follows: find $y \in \boldsymbol{V}$ solution of

$$\alpha\int_\Omega y\cdot w + \nu\int_\Omega \nabla y\cdot\nabla w + \int_\Omega y\cdot\nabla y\cdot w = < f, w >_{H^{-1}(\Omega)^d\times H_0^1(\Omega)^d} + \alpha\int_\Omega g\cdot w, \quad \forall w \in \boldsymbol{V}. \tag{2.3}$$

The following results holds true:

Proposition 2.2. *a) Assume $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz. There exists a least one solution y of (2.3) satisfying*

$$\alpha\|y\|_2^2 + \nu\|\nabla y\|_2^2 \leq \frac{\alpha}{\nu}\left(\nu\|g\|_2^2 + \frac{c_0}{\alpha}\|f\|_{H^{-1}(\Omega)^d}^2\right) \tag{2.4}$$

where $c_0 > 0$, only connected to the Poincaré constant, dependent on Ω. If moreover, Ω is C^2 and $f \in L^2(\Omega)^d$, then $y \in H^2(\Omega)^d \cap \boldsymbol{V}$.

b) Let us define $Q(g,f,\alpha,\nu)$ as follows :

$$Q(g,f,\alpha,\nu)=\begin{cases}\dfrac{1}{\nu^3}\left(\nu\|g\|_2^2+\dfrac{c_0}{\alpha}\|f\|^2_{H^{-1}(\Omega)^2}\right), & \text{if} \quad d=2,\\ \dfrac{\alpha^{1/2}}{\nu^{7/2}}\left(\nu\|g\|_2^2+\dfrac{c_0}{\alpha}\|f\|^2_{H^{-1}(\Omega)^3}\right), & \text{if} \quad d=3.\end{cases} \tag{2.5}$$

If $Q(g,f,\alpha,\nu)$ is small enough, then the solution of (2.3) is unique.

Proof. The point a) is well-known and we refer to [13]. Let now $y_1\in \boldsymbol{V}$ and $y_2\in \boldsymbol{V}$ be two solutions of (2.3). Set $Y=y_1-y_2$. Then,

$$\alpha\int_\Omega Y\cdot w+\nu\int_\Omega \nabla Y\cdot\nabla w+\int_\Omega y_2\cdot\nabla Y\cdot w+\int_\Omega Y\cdot\nabla y_1\cdot w=0 \qquad \forall w\in \boldsymbol{V}.$$

We now take $w=Y$ and use that $\int_\Omega y_2\cdot\nabla Y\cdot Y=0$. If $d=2$, we get

$$\begin{aligned}\alpha\int_\Omega |Y|^2+\nu\int_\Omega|\nabla Y|^2&=-\int_\Omega Y\cdot\nabla y_1\cdot Y\le c\|Y\|_2\|\nabla y_1\|_2\|\nabla Y\|_2\\ &\le \alpha\|Y\|_2^2+\frac{c}{\alpha}\|\nabla Y\|_2^2\|\nabla y_1\|_2^2\end{aligned}$$

leading to $(\nu-\frac{c}{\alpha}\|\nabla y_1\|_2^2)\|\nabla Y\|_2^2\le 0$. Consequently, if $\nu\|g\|_2^2+\frac{c_0}{\alpha}\|f\|^2_{H^{-1}(\Omega)^d}<c_1\nu^3$ for some $c_1>0$, then $\|\nabla Y\|_2^2\le 0$ and eventually $Y=0$. On the other hand, if $d=3$, we obtain

$$\begin{aligned}\alpha\int_\Omega |Y|^2+\nu\int_\Omega|\nabla Y|^2&=-\int_\Omega Y\cdot\nabla y_1\cdot Y\le c\|Y\|_4^2\|\nabla y_1\|_2\le c\|Y\|_2^{\frac12}\|\nabla Y\|_2^{\frac32}\|\nabla y_1\|_2\\ &\le \alpha\|Y\|_2^2+\frac{c}{\alpha^{1/3}}\|\nabla Y\|_2^2\|\nabla y_1\|_2^{\frac43}\end{aligned}$$

leading to $(\nu-\frac{c}{\alpha^{1/3}}\|\nabla y_1\|_2^{\frac43})\|\nabla Y\|_2^2\le 0$. Consequently, if $\nu\|g\|_2^2+\frac{c_0}{\alpha}\|f\|^2_{H^{-1}(\Omega)^d}<c_1\nu^{7/2}\alpha^{-1/2}$ for some $c_1>0$, then $\|\nabla Y\|_2^2\le 0$ and again $Y=0$. □

We now introduce our least-squares functional $E:\boldsymbol{V}\to\mathbb{R}^+$ as follows

$$E(y):=\frac12\int_\Omega(\alpha|v|^2+|\nabla v|^2) \tag{2.6}$$

where the corrector $v\in\boldsymbol{V}$ is the unique solution of

$$\begin{aligned}\alpha\int_\Omega v\cdot w+\int_\Omega\nabla v\cdot\nabla w=&-\alpha\int_\Omega y\cdot w-\nu\int_\Omega\nabla y\cdot\nabla w-\int_\Omega y\cdot\nabla y\cdot w\\ &+<f,w>_{H^{-1}(\Omega)^d\times H_0^1(\Omega)^d}+\alpha\int_\Omega g\cdot w,\quad\forall w\in\boldsymbol{V}.\end{aligned} \tag{2.7}$$

The infimum of E is equal to zero and is reached by a solution of (2.3). In this sense, the functional E is a so-called error functional which measures, through the corrector variable v, the deviation of the pair y from being a solution of the underlying equation (2.3).

Beyond this statement, we would like to argue why we believe it is a good idea to use a (minimization) least-squares approach to approximate the solution of (2.3) by minimizing the functional E. Our main result of this section is a follows:

Theorem 2.3. *Assume that $Q(g,f,\alpha,\nu)$ is small enough. There is a positive constant C, such that if $\{y_k\}_{k>0}$ is a sequence in*

$$\mathbb{B}:=\{y\in\boldsymbol{V}:\|y\|_{H_0^1(\Omega)^d}\le C\}$$

with $E'(y_k)\to 0$ as $k\to\infty$, then the whole sequence $\{y_k\}_{k\in\mathbb{N}}$ converges strongly as $k\to\infty$ in $\boldsymbol{V}$ to a solution $\overline{y}$ of (2.3).

As in [11], we divide the proof in two main steps.

(1) First, we use a typical a priori bound to show that leading the error functional E down to zero implies strong convergence to the unique solution of (2.3).
(2) Next, we show that taking the derivative E' to zero actually suffices to take E to zero.

Proposition 2.4. *Assuming that $Q(g, f, \alpha, \nu)$ is small enough and let $\overline{y}$ be a solution of (2.3). For every $y \in \boldsymbol{V}$, we have*

$$\|y - \overline{y}\|_{H_0^1(\Omega)^d} \leq 2\,\nu^{-1}\sqrt{E(y)}. \tag{2.8}$$

The control of the norm $\|y-\overline{y}\|_{H_0^1(\Omega)^d}$ is not uniform with respect to ν, in agreement with the behavior of the solution of (1.3) (see notably (2.4)) as ν goes to zero. Moreover, this proposition very clearly establishes that as we take down the error E to zero, we get closer, in the strong norm, to the solution of the problem, and so, it justifies why a promising strategy to find good approximations of the solution of problem (2.3) is to look for global minimizers of the extremal problem:

$$\inf_{y \in \boldsymbol{V}} E(y). \tag{2.9}$$

Proof. The proof of Proposition 2.4 basically amounts to a typical a priori estimate which is essentially the same that the proof of uniqueness in page 112 in [20]. For any $y \in \boldsymbol{V}$, let v be the corresponding corrector and let $Y = y - \overline{y}$. We have

$$\alpha \int_\Omega Y \cdot w + \nu \int_\Omega \nabla Y \cdot \nabla w + \int_\Omega y \cdot \nabla Y \cdot w + \int_\Omega Y \cdot \nabla \overline{y} \cdot w = -\alpha \int_\Omega v \cdot w - \int_\Omega \nabla v \cdot \nabla w \qquad \forall w \in \boldsymbol{V}. \tag{2.10}$$

For $w = Y$, this equality rewrites

$$\alpha \int_\Omega |Y|^2 + \nu \int_\Omega |\nabla Y|^2 = -\int_\Omega Y \cdot \nabla \overline{y} \cdot Y - \alpha \int_\Omega v \cdot Y - \int_\Omega \nabla v \cdot \nabla Y.$$

Moreover,

$$\Big| \int_\Omega Y \cdot \nabla \overline{y} \cdot Y \Big| \leq c \|Y\|_2 \|\nabla \overline{y}\|_2 \|\nabla Y\|_2 \leq \frac{c}{\nu} \|Y\|_2^2 \|\nabla \overline{y}\|_2^2 + \frac{\nu}{4} \|\nabla Y\|_2^2,$$

if $d = 2$ and

$$\Big| \int_\Omega Y \cdot \nabla \overline{y} \cdot Y \Big| \leq c \|Y\|_2^{\frac{1}{2}} \|\nabla \overline{y}\|_2 \|\nabla Y\|_2^{\frac{3}{2}} \leq \frac{c}{\nu^3} \|Y\|_2^2 \|\nabla \overline{y}\|_2^4 + \frac{\nu}{4} \|\nabla Y\|_2^2,$$

if $d = 3$. Consequently, the inequalities

$$\Big| \alpha \int_\Omega v \cdot Y \Big| \leq \frac{\alpha}{2} \int_\Omega |v|^2 + \frac{\alpha}{2} \int_\Omega |Y|^2, \quad \Big| \int_\Omega \nabla v \cdot \nabla Y \Big| \leq \frac{1}{\nu} \int_\Omega |\nabla v|^2 + \frac{\nu}{4} \int_\Omega |\nabla Y|^2$$

lead to the estimate

$$(\alpha - \frac{c}{\nu} \|\nabla \overline{y}\|_2^2) \int_\Omega |Y|^2 + \nu \int_\Omega |\nabla Y|^2 \leq \alpha \int_\Omega |v|^2 + \frac{2}{\nu} \int_\Omega |\nabla v|^2$$

if $d = 2$ and to the estimate

$$(\alpha - \frac{c}{\nu^3} \|\nabla \overline{y}\|_2^4) \int_\Omega |Y|^2 + \nu \int_\Omega |\nabla Y|^2 \leq \alpha \int_\Omega |v|^2 + \frac{2}{\nu} \int_\Omega |\nabla v|^2$$

if $d = 3$.

Eventually, if $d = 2$ and if there exists $c > 0$ such that if $\nu \|g\|_2^2 + \frac{c_0}{\alpha} \|f\|_{H^{-1}(\Omega)^2}^2 \leq c\nu^3$, we deduce that

$$\int_\Omega |\nabla Y|^2 \leq 2\nu^{-1} \max(1, 2\nu^{-1}) E(y).$$

If $d = 3$, the same conclusion holds true if there exists $c > 0$ such that if $\nu \|g\|_2^2 + \frac{c_0}{\alpha} \|f\|_{H^{-1}(\Omega)^3}^2 \leq c\nu^{7/2}\alpha^{-1/2}$. □

A practical way of taking a functional to its minimum is through some (clever) use of descent directions, i.e. the use of its derivative. In doing so, the presence of local minima is always something that may dramatically spoil the whole scheme. The unique structural property that discards this possibility is the strict convexity of the functional. However, for non-linear equations like (2.3), one cannot expect this property to hold for the functional E in (2.6). Nevertheless, we insist in that for a descent strategy applied to our extremal problem (2.9), numerical procedures cannot converge except to a global minimizer leading E down to zero. In doing so, thanks to Proposition 2.4, we are establishing the strong convergence of approximations to the unique solution of (2.3).

Indeed, we would like to show that the only critical points for E correspond to solutions of (2.3). In such a case, the search for an element y solution of (2.3) is reduced to the minimization of E, as indicated in the preceding paragraph.

For any $y \in \boldsymbol{V}$, we now look for an element $Y_1 \in \boldsymbol{V}$ solution of the following formulation

$$\alpha \int_\Omega Y_1 \cdot w + \nu \int_\Omega \nabla Y_1 \cdot \nabla w + \int_\Omega (y \cdot \nabla Y_1 + Y_1 \cdot \nabla y) \cdot w = -\alpha \int_\Omega v \cdot w - \int_\Omega \nabla v \cdot \nabla w, \forall w \in \boldsymbol{V} \tag{2.11}$$

where $v \in \boldsymbol{V}$ is the corrector (associated to y) solution of (2.7). Y_1 enjoys the following property.

Proposition 2.5. *There exists $c > 0$ such that, for all $y \in \boldsymbol{V}$ satisfying $\frac{1}{\nu\alpha}\|\nabla y\|_2^2 < c$ if $d = 2$ and $\frac{1}{\nu^3\alpha}\|\nabla y\|_2^4 < c$ if $d = 3$, there exists a unique solution Y_1 of (2.11) associated to y. This solution satisfies*

$$\|Y_1\|_{\boldsymbol{V}} \leq M$$

for some constant $M > 0$, independent of y.

Proof. We define the bilinear and continuous form $a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ by

$$a(Y, w) = \alpha \int_\Omega Y \cdot w + \nu \int_\Omega \nabla Y \cdot \nabla w + \int_\Omega (y \cdot \nabla Y + Y \cdot \nabla y) \cdot w. \tag{2.12}$$

Let $d = 2$. Using similar computations than previously, we get that

$$a(Y, Y) \geq (\alpha - \frac{c}{\nu}\|\nabla y\|_2^2) \int_\Omega |Y|^2 + \frac{\nu}{2} \int_\Omega |\nabla Y|^2, \quad \forall Y \in \boldsymbol{V}, \tag{2.13}$$

for some constant $c > 0$. Lax-Milgram lemma leads to the existence and uniqueness of Y_1 provided $\|\nabla y\|_2$ is small enough. Then, putting $w = Y_1$ in (2.11) implies

$$\alpha \int_\Omega |Y_1|^2 + \int_\Omega |\nabla Y_1|^2 = -\int_\Omega Y_1 . \nabla y . Y_1 - \alpha \int_\Omega v . Y_1 - \int_\Omega \nabla v . \nabla Y_1$$

and therefore

$$(\alpha - \frac{c}{\nu}\|\nabla y\|_2^2) \int_\Omega |Y_1|^2 + \frac{\nu}{2} \int_\Omega |\nabla Y_1|^2 \leq \frac{\alpha}{2} \int_\Omega |v|^2 + \frac{1}{\nu} \int_\Omega |\nabla v|^2 \tag{2.14}$$

so that $\|Y_1\|_{\boldsymbol{V}}^2 \leq 2\nu^{-2} E(y)$. On the other hand, $w = v$ in (2.7) leads to

$$\begin{aligned}
\int_\Omega \alpha|v|^2 + |\nabla v|^2 =& -\alpha \int_\Omega y \cdot v - \nu \int_\Omega \nabla y \cdot \nabla v - \int_\Omega y \cdot \nabla y \cdot v + \langle f, v \rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} + \alpha \int_\Omega g \cdot v \\
\leq& \frac{\alpha}{4} \int_\Omega |v|^2 + \alpha \int_\Omega |y|^2 + \nu^2 \int_\Omega |\nabla y|^2 + \frac{1}{4} \int_\Omega |\nabla v|^2 + \frac{1}{4} \int_\Omega |\nabla v|^2 + c_1 \Big(\int_\Omega |\nabla y|^2 \Big)^2 \\
&+ c_0 \|f\|_{H^{-1}(\Omega)^2}^2 + \frac{1}{4} \int_\Omega |\nabla v|^2 + \frac{\alpha}{4} \int_\Omega |v|^2 + \alpha \int_\Omega |g|^2
\end{aligned}$$

for some positive constant c_1, and therefore, after some re-ordering,

$$\alpha \int_\Omega |v|^2 + \int_\Omega |\nabla v|^2 \leq 2\alpha \int_\Omega |y|^2 + 2\nu^2 \int_\Omega |\nabla y|^2 + c \Big(\int_\Omega |\nabla y|^2 \Big)^2 + 2c_0 \|f\|_{H^{-1}(\Omega)^2}^2 + 2\alpha \int_\Omega |g|^2$$

leading to

$$E(y) \leq \left(c_p^2 \alpha + \nu^2 + \frac{c}{2} \|y\|_{\boldsymbol{V}}^2 \right) \|y\|_{\boldsymbol{V}}^2 + c_0 \|f\|_{H^{-1}(\Omega)^2}^2 + \alpha \|g\|_2^2$$

where c_p is the Poincaré constant. This inequality coupled with (2.14) or (2.15) and the hypothesis on the size of $\|\nabla y\|_2$ implies our statement. Precisely,

$$\|Y_1\|_{\boldsymbol{V}}^2 \leq \left(c_p^2\alpha + \nu^2 + \frac{c}{2}\nu\alpha\right)\nu^{-1}\alpha + c_0\nu^{-1}\|f\|_{H^{-1}(\Omega)^2}^2 + \alpha\nu^{-1}\|g\|_2^2.$$

The case $d = 3$ for which (2.14) is replaced by

$$(\alpha - \frac{c}{\nu^3}\|\nabla y\|_2^4)\int_\Omega |Y_1|^2 + \frac{\nu}{2}\int_\Omega |\nabla Y_1|^2 \leq \frac{\alpha}{2}\int_\Omega |v|^2 + \frac{1}{\nu}\int_\Omega |\nabla v|^2 \tag{2.15}$$

is similar. □

We are now in position to prove the following result

Proposition 2.6. *There exists a positive constant C such that if $\{y_k\}_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ defined by $\mathbb{B} = \{y \in \boldsymbol{V} : \frac{1}{\nu\alpha}\|\nabla y\|_2^2 < C\}$ if $d = 2$ and $\mathbb{B} = \{y \in \boldsymbol{V} : \frac{1}{\nu^3\alpha}\|\nabla y\|_2^4 < C\}$ if $d = 3$ with $E'(y_k) \to 0$ as $k \to \infty$, then $E(y_k) \to 0$ as $k \to \infty$.*

The condition on the size of y in this statement is coherent with our hypotheses because the norm $\|\overline{y}\|_{\boldsymbol{V}}$ of $\overline{y}$ in bounded in term of Q, assumed small: precisely, estimate (2.4) implies that $\alpha\|y\|_2^2 + \nu\|\nabla y\|_2^2$ is bounded by $\alpha\nu^2\, Q(g, f, \alpha, \nu)$ if $d = 2$ and by $\alpha^{1/2}\nu^{5/2}Q(g, f, \alpha, \nu)$ if $d = 3$.

Proof. The error functional E is differentiable as functional defined on the Hilbert space $\boldsymbol{V}$, because the operator $y \to v$ taking each $y \in \boldsymbol{V}$ into its associated corrector v, as stated above is a differentiable operation. Indeed, $E'(y)$ can always be identified with an element of $\boldsymbol{V}$ itself. For any $Y \in \boldsymbol{V}$, we have

$$E'(y) \cdot Y = \int_\Omega \alpha\, v \cdot V + \nabla v \cdot \nabla V \tag{2.16}$$

where $V \in \boldsymbol{V}$ is the unique solution of

$$\alpha\int_\Omega V \cdot w + \int_\Omega \nabla V \cdot \nabla w = -\alpha\int_\Omega Y \cdot w - \nu\int_\Omega \nabla Y \cdot \nabla w - \int_\Omega (y \cdot \nabla Y + Y \cdot \nabla y) \cdot w, \forall w \in \boldsymbol{V}. \tag{2.17}$$

In particular, taking $Y = Y_1$ defined by (2.11), we easily check that

$$E'(y) \cdot Y_1 = \int_\Omega \alpha|v|^2 + |\nabla v|^2 = 2E(y), \quad \forall y \in \boldsymbol{V}. \tag{2.18}$$

Let now, for any $k \in \mathbb{N}$, $Y_{1,k}$ be the solution of (2.11) associated to y_k. The previous equality writes $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$ and implies our statement, since from Proposition 2.5, $Y_{1,k}$ is uniformly bounded in $\boldsymbol{V}$. □

Eventually, Theorem 2.3 follows from Proposition 2.4 and Proposition 2.6 with $\mathbb{B} = (\nu\alpha)^{1/2}B_{\boldsymbol{V}}(0, c)$ if $d = 2$ and $\mathbb{B} = (\nu^3\alpha)^{1/4}B_{\boldsymbol{V}}(0, c)$ if $d = 3$.

Theorem 2.3 is a general convergence result that do not take into account the particular method to produce such sequence $\{y_k\}_{(k\in\mathbb{N})}$ with $E'(y_k) \to 0$. In practice, however, one would typically use a gradient method to calculate iteratively such sequences. The following lemma ensures that a gradient method for the functional E in (3.1) will always converge to the solution of (2.3), provided the initial guess y_0 of the minimizing sequence is sufficiently close to the solution $\overline{y}$ of (2.3).

Proposition 2.7. *Assume that $Q(g, f, \alpha, \nu)$ is small enough. There is a known, specific positive constant $C = C(\|\overline{y}\|_{\boldsymbol{V}}, \alpha, \nu)$ such that if $\|y_0 - \overline{y}\|_{\boldsymbol{V}} < C$, then a gradient based method for E starting from y_0 will always converge to $\overline{y}$.*

Proof. Let $(y_k)_{k\in\mathbb{N}}$ a minimizing sequence for E based on the gradient E', i.e. $(y_{k+1}-y_k,w) = -\lambda E'(y_k)\cdot w$, for all $w\in \boldsymbol{V}$ and $\lambda>0$. We note by g_k the element of $\boldsymbol{V}$ defined by $(g_k,w)_{\boldsymbol{V}} = E'(y_k)\cdot w$ for all $w\in\boldsymbol{V}$, equivalently

$$(g_k,w)_{\boldsymbol{V}} = -\int_\Omega \alpha v_k\cdot w + \nu\nabla v_k\cdot\nabla w + (w\cdot\nabla y_k + y_k\cdot\nabla w)\cdot v_k, \forall w\in\boldsymbol{V}. \tag{2.19}$$

v_k is the corrector associated to y_k. In particular, we check that $\|g_k\|_{\boldsymbol{V}}$ is uniformly bounded as soon as y_k is uniformly bounded.

We first check that the following equality holds true:

$$\|y_{k+1}-\overline{y}\|^2_{\boldsymbol{V}} - \|y_k-\overline{y}\|^2_{\boldsymbol{V}} = 2\lambda E'(y_k)\cdot(\overline{y}-y_k) + \lambda^2\|g_k\|^2_{\boldsymbol{V}}, \quad \forall k\in\mathbb{N}. \tag{2.20}$$

Indeed, we have

$$\begin{aligned}
-\lambda E'(y_k)\cdot(\overline{y}-y_k) &= (y_{k+1}-y_k,\overline{y}-y_k)\\
&= -\|y_{k+1}-\overline{y}\|^2_{\boldsymbol{V}} + \|y_k-\overline{y}\|^2_{\boldsymbol{V}} + (y_{k+1}-\overline{y},y_{k+1}-y_k)\\
&= -\|y_{k+1}-\overline{y}\|^2_{\boldsymbol{V}} + \|y_k-\overline{y}\|^2_{\boldsymbol{V}} - \lambda E'(y_k)\cdot(y_{k+1}-\overline{y})\\
&= -\|y_{k+1}-\overline{y}\|^2_{\boldsymbol{V}} + \|y_k-\overline{y}\|^2_{\boldsymbol{V}} - \lambda E'(y_k)\cdot(y_{k+1}-y_k) - \lambda E'(y_k)\cdot(y_k-\overline{y})
\end{aligned}$$

and the equality (2.20) follows since the third term (in the right hand side) is

$$-\lambda E'(y_k)\cdot(y_{k+1}-y_k) = -\lambda(g_k,y_{k+1}-y_k) = \lambda^2 E'(y_k)\cdot g_k = \lambda^2\|g_k\|^2.$$

The strategy is then to show that the quantity $E'(y_0)\cdot(\overline{y}-y_0)$ becomes non-positive, if the initial guess y_0 is sufficiently close, in a precise quantitative way, to the exact solution $\overline{y}$. Taking $\lambda>0$ small enough, it will follows from (2.20), that if y_0 belongs to the ball $\mathbb{B}$ of Proposition 2.6, then, recursively, every element of the sequence $\{y_k\}_{k\in\mathbb{N}}$ will stay in $\mathbb{B}$.

Let y_0 be an arbitrary field in $\boldsymbol{V}$, and recall formula (2.19) for the derivative of E at y_0, applied to the difference $Y=\overline{y}-y_0$

$$E'(y_0)\cdot Y = -\int_\Omega \alpha v_0\cdot Y + \nu\nabla v_0\cdot\nabla Y + (Y\cdot\nabla y_0 + y_0\cdot\nabla Y)\cdot v_0$$

where v_0 is the corrector associated with y_0. On the other hand, using v_0 as a test function in (2.10) for $y=y_0$ (which is the difference of the equations for y_0 with its corrector v_0 and for the exact solution $\overline{y}$), it is a matter of some careful algebra to arrive at

$$\begin{aligned}
E'(y_0)\cdot Y &= -\int_\Omega \alpha|v_0|^2 + |\nabla v_0|^2 + \int_\Omega Y\cdot\nabla Y\cdot v_0 = -2E(y_0) + \int_\Omega Y\cdot\nabla Y\cdot v_0.\\
&\le -2E(y_0) + c\|Y\|_2\|\nabla Y\|_2\|\nabla v_0\|_2
\end{aligned} \tag{2.21}$$

assuming $d=2$ in the last line.We now take into account Proposition 2.4 so that $\|Y\|^2_{H^1_0(\Omega)^2} \le c^2\nu^{-2}E(y_0)$, for some constant c provided $Q(g,f,\alpha,\nu)$ is small enough leading to the estimate

$$E'(y_0)\cdot Y \le -2E(y_0) + c\nu^{-2}E(y_0)^{3/2},$$

for a certain known constant c independent of y_0, α and ν. If we can ensure that the size of the corrector v_0 is such that $\sqrt{E(y_0)} < 2c^{-1}\nu^2$, whenever $\|Y\|_{H^1_0(\Omega)^2} < c_1$ for some positive known constant c_1, then we would indeed have $E'(y_0)\cdot Y<0$. This sign condition is informing us that the flow of E is always pointing inwards in the ball determined $B_{\boldsymbol{V}}(0,c_1)$. If we take c_1 even smaller if necessary to guarantee that $\|Y\|_{H^1_0(\Omega)^2} < c_1$ implies $y\in\mathbb{B}$ where $\mathbb{B}$ is the ball, centered at zero, in Proposition 2.6, then we would have that all integral curves starting under the condition $y_0\in B_{\boldsymbol{V}}(\overline{y},c_1)$ will converge to $\overline{y}$ since in this ball there cannot be critical points of E other than $\overline{y}$ itself, according to Theorem 2.3. It remains, hence, to quantify the continuity of E at the solution $\overline{y}$. To check this, we use again v_0 as a test function in (2.10) and obtain

$$\alpha\|v_0\|_2^2 + \|\nabla v_0\|_2^2 = \alpha\int_\Omega Yv_0 + \nu\int_\Omega\nabla Y\cdot\nabla v_0 + \int(\overline{y}\cdot\nabla Y + Y\cdot\nabla\overline{y} - Y\cdot\nabla Y)\cdot v_0 \tag{2.22}$$

Since

$$\int Y \cdot \nabla Y \cdot v_0 \leq \|Y\|_2 \|\nabla Y\|_2 \|\nabla v_0\|_2 \leq c\|Y\|^2_{H^1_0(\Omega)^2} \|\nabla v_0\|_2,$$

and

$$\int \overline{y} \cdot \nabla Y \cdot v_0 \leq \|\overline{y}\|_2 \|\nabla Y\|_2 \|\nabla v_0\|_2 \leq \|\overline{y}\|_2 \|Y\|_{\boldsymbol{V}} \|\nabla v_0\|_2,$$

and

$$\int Y \cdot \nabla \overline{y} \cdot v_0 \leq \|Y\|_2 \|\nabla \overline{y}\|_2 \|\nabla v_0\|_2 \leq c\|\nabla \overline{y}\|_2 \|Y\|_{\boldsymbol{V}} \|\nabla v_0\|_2,$$

we find

$$\sqrt{E(y_0)} \leq c\big(\|Y\|_{\boldsymbol{V}} + \|\overline{y}\|_2 + \|\nabla \overline{y}\|_2\big)\|Y\|_{\boldsymbol{V}} \tag{2.23}$$

for some $c > 0$ independent of y_0, α and ν. Remark that this inequality is the reverse of (2.8). We clearly see that we can make the left-hand side small by making the right-hand side small in a quantified way. Precisely, we easily check that the inequality $\sqrt{E(y_0)} < 2c^{-1}\nu^2$ holds true as soon as

$$\|y_0 - \overline{y}\|_{\boldsymbol{V}} \leq c\left(\sqrt{\|\overline{y}\|^2_{\boldsymbol{V}} + 4\nu^2} - \|\overline{y}\|_{\boldsymbol{V}}\right)$$

where the constant $c > 0$ related to the Poincaré inequality depends only on Ω. Remark that $\sqrt{\|\overline{y}\|^2_{\boldsymbol{V}} + 4\nu^2} - \|\overline{y}\|_{\boldsymbol{V}} \leq 2\nu$.

The case $d = 3$ is similar. □

From a purely practical standpoint, however, checking "a posteriori" computed iterates y_k will tell us whether we are getting close to the unique solution $\overline{y}$, because numbers $E(y_k)$ become steadily and virtually zero, or they stay bounded away from zero.

Remark 2.8. *Section 6, chapter IV of the book* [4] *introduces a least-squares method in order to solve an Oseen type equation (without incompressibility constraint). The convergence of any minimizing sequence toward a solution y is proved under the condition that the operator $DF(y)$ defined as follows*

$$DF(y) \cdot w = c\, w - \nu\Delta w + [(w \cdot \nabla)y + (y \cdot \nabla)w], \quad \forall w \in \boldsymbol{V} \tag{2.24}$$

(for some $c > 0$) is an isomorphism from $\boldsymbol{V}$ onto $\boldsymbol{V}'$. This property is actually necessary to determine recursively in a unique way a (minimizing) sequence from its first element (see (2.19)). The smallness assumption on $Q(g, f, \alpha, \nu)$ is a sufficient condition for the operator $DF(\overline{y})$ (with $c = \alpha$ in (2.24)) to be an isomorphism. Moreover, it appears that this assumption implies the uniqueness of $\overline{y}$, the solution we are looking for. As far as we know, determine a weaker condition ensuring that $DF(\overline{y})$ is an isomorphism is an open question.

Remark 2.9. *The error functional E in* [2] *is defined in a slightly different way, precisely $E(y) = \frac{1}{2}\int_\Omega \alpha|v|^2 + \nu|\nabla v|^2$, making appear explicitly the parameter ν. The term $\int_\Omega \nabla v \cdot \nabla w$ in (2.7) defining the corrector is then replaced by $\int_\Omega \nu \nabla v \cdot \nabla w$. Our analysis remains true in this situation.*

3. Application to the Implicit Euler Scheme

As done in [2], one may use the least-squares method analyzed in the previous section to solve the implicit scheme given by (1.2). For simplicity, we take $\theta = 1$ and write y^n for y^n_θ. According to the previous section, in order to compute y^{n+1} from y^n, one may consider the following extremal problem

$$\inf_{y \in \boldsymbol{V}} E_n(y), \quad E_n(y) = \frac{1}{2}\int_\Omega \alpha|v|^2 + |\nabla v|^2 \tag{3.1}$$

where the corrector $v \in \boldsymbol{V}$ solves

$$\alpha \int_\Omega v \cdot w + \int_\Omega \nabla v \cdot \nabla w = -\alpha \int_\Omega y \cdot w - \nu \int_\Omega \nabla y \cdot \nabla w - \int_\Omega y \cdot \nabla y \cdot w + < f^n, w >_{H^{-1}(\Omega)^d \times H^1_0(\Omega)^d} + \alpha \int_\Omega g^n \cdot w, \quad \forall w \in \boldsymbol{V} \tag{3.2}$$

where α, g^n and f^n are given by (1.4). The natural choice is to initialize the minimizing sequence, says $(y_k^{n+1})_{k\in\mathbb{N}}$ for E_n with the element $g^n = y^n$, i.e. $y_0^{n+1} = y^n$. According to Proposition 2.7, if y^{n+1} is close enough to y^n for the $\boldsymbol{V}$-norm, then the strong convergence of the minimizing sequence $(y_j^n)_{j\in\mathbb{N}}$ to y^{n+1} holds true. One main goal in this section is to check that two consecutive elements of the sequence y^n defined by recurrence from the scheme (1.2) are close each other as soon as δt, the time step, is small enough. Before to give such result in Theorem 3.3, we state several intermediate results.

Proposition 3.1. *Let $d = 2$ or $d = 3$, $(f^n)_{n\in\mathbb{N}}$ a sequence in $H^{-1}(\Omega)^d$, $\alpha > 0$ and $y^0 \in L^2(\Omega)^d$.*

a) We define by recurrence for all $n \in \mathbb{N}$, $y^{n+1} \in \boldsymbol{V}$, as solution of

$$\alpha \int_\Omega (y^{n+1} - y^n) \cdot w + \nu \int_\Omega \nabla y^{n+1} \cdot \nabla w + \int_\Omega y^{n+1} \cdot \nabla y^{n+1} \cdot w = < f^n, w >_{H^{-1}(\Omega)^d \times H^1_0(\Omega)^d} \tag{3.3}$$

for all $w \in \boldsymbol{V}$. For all $n \in \mathbb{N}$, y^{n+1} satisfies

$$\alpha \|y^{n+1}\|_2^2 + \nu \|\nabla y^{n+1}\|_2^2 \le \frac{c_0}{\nu} \|f^n\|^2_{H^{-1}(\Omega)^d} + \alpha \|y^n\|_2^2 \tag{3.4}$$

where $c > 0$ depends only on Ω and for all $n \in \mathbb{N}^\star$:

$$\|y^n\|_2^2 + \frac{\nu}{\alpha} \sum_{k=1}^{n} \|\nabla y^k\|_2^2 \le \frac{1}{\nu} \Big(\frac{c_0}{\alpha} \sum_{k=0}^{n-1} \|f^k\|^2_{H^{-1}(\Omega)^d} + \nu \|y^0\|_2^2 \Big). \tag{3.5}$$

b) Let us define $\mathcal{M}(f, \alpha, \nu)$ as follows:

$$\mathcal{M}(f,\alpha,\nu) = \begin{cases} \dfrac{1}{\nu^3} \Big(\dfrac{c_0}{\alpha} \displaystyle\sum_{k=0}^{n-1} \|f^k\|^2_{H^{-1}(\Omega)^2} + \nu \|y^0\|_2^2 \Big), & \text{if} \quad d = 2, \\ \dfrac{\alpha^{1/2}}{\nu^{7/2}} \Big(\dfrac{c_0}{\alpha} \displaystyle\sum_{k=0}^{n-1} \|f^k\|^2_{H^{-1}(\Omega)^3} + \nu \|y^0\|_2^2 \Big), & \text{if} \quad d = 3. \end{cases} \tag{3.6}$$

If $\mathcal{M}(f, \alpha, \nu)$ is small enough, then the solution of (3.3) is unique.

Proof.

(a) The existence of y^{n+1} is given in Proposition 2.2. Taking $w = y_{n+1}$ in (3.3) leads to

$$\alpha \int_\Omega |y^{n+1}|^2 + \nu \int_\Omega |\nabla y^{n+1}|^2 + \int_\Omega y^{n+1} \cdot \nabla y^{n+1} \cdot y^{n+1} = \langle f^n, y^{n+1} \rangle_{H^{-1}(\Omega)^d \times H^1_0(\Omega)^d} + \alpha \int_\Omega y^n \cdot y^{n+1}$$

and since $\int_\Omega y^{n+1} \cdot \nabla y^{n+1} \cdot y^{n+1} = 0$, we obtain

$$\begin{aligned} \alpha \int_\Omega |y^{n+1}|^2 + \nu \int_\Omega |\nabla y^{n+1}|^2 &\le \sqrt{c_0} \|f^n\|_{H^{-1}(\Omega)^d} \|\nabla y^{n+1}\|_2 + \alpha \|y^n\|_2 \|y^{n+1}\|_2 \\ &\le \frac{c_0}{2\nu} \|f^n\|^2_{H^{-1}(\Omega)^d} + \frac{\nu}{2} \|\nabla y^{n+1}\|_2^2 + \frac{\alpha}{2} \|y^n\|_2^2 + \frac{\alpha}{2} \|y^{n+1}\|_2^2. \end{aligned}$$

This shows (3.4). Summing in n easily gives (3.5).

(b) Let $n \in \mathbb{N}$ and let $y_1^{n+1}, y_2^{n+1} \in \boldsymbol{V}$ be two solutions of (3.3). Let $Y = y_1^{n+1} - y_2^{n+1}$. Then,

$$\alpha \int_\Omega Y \cdot w + \nu \int_\Omega \nabla Y \cdot \nabla w + \int_\Omega y_2^{n+1} \cdot \nabla Y \cdot w + \int_\Omega Y . \nabla y_1^{n+1} . w = 0 \qquad \forall w \in \boldsymbol{V}.$$

and then, for $w = Y$ (using that $\int_\Omega y_2^{n+1}.\nabla Y.Y = 0$) and for $d = 2$

$$\alpha\int_\Omega |Y|^2 + \nu\int_\Omega |\nabla Y|^2 = -\int_\Omega Y.\nabla y_1^{n+1}.Y \le c\|Y\|_2\|\nabla Y\|_2\|\nabla y_1^{n+1}\|_2$$
$$\le \alpha\|Y\|_2^2 + \frac{c}{\alpha}\|\nabla Y\|_2^2\|\nabla y_1^{n+1}\|_2^2.$$

It follows that $(\nu - \frac{c}{\alpha}\|\nabla y_1^{n+1}\|_2^2)\|\nabla Y\|_2^2 \le 0$. Moreover, in view of (3.5),

$$\frac{\nu}{\alpha}\|\nabla y_1^{n+1}\|_2^2 \le \frac{1}{\nu}\Big(\frac{c_0}{\alpha}\sum_{k=0}^{n}\|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu\|y^0\|_2^2\Big).$$

Therefore, if $\frac{c_0}{\alpha}\sum_{k=0}^{n}\|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu\|y_0\|_2^2 < c_1\nu^3$ for some constant $c_1 > 0$, then $\|\nabla Y\|_2^2 \le 0$.

Similarly, if $d = 3$, we can write

$$\alpha\int_\Omega |Y|^2 + \nu\int_\Omega |\nabla Y|^2 = -\int_\Omega Y.\nabla y_1^{n+1}.Y \le c\|Y\|_2^{\frac{1}{2}}\|\nabla Y\|_2^{\frac{3}{2}}\|\nabla y_1^{n+1}\|_2$$
$$\le \alpha\|Y\|_2^2 + \frac{c}{\alpha^{1/3}}\|\nabla Y\|_2^2\|\nabla y_1^{n+1}\|_2^{\frac{4}{3}}$$

leading to $(\nu - \frac{c}{\alpha^{1/3}}\|\nabla y_1^{n+1}\|_2^{\frac{4}{3}})\|\nabla Y\|_2^2 \le 0$. Arguing as before, if $\frac{c_0}{\alpha}\sum_{k=0}^{n}\|f^k\|_{H^{-1}(\Omega)^3}^2 + \nu\|y^0\|_2^2 < c_1\alpha^{-\frac{1}{2}}\nu^{\frac{7}{2}}$ for some $c_1 > 0$, then $\|\nabla Y\|_2^2 \le 0$. □

Proposition 3.2. *Assume hypotheses of Proposition 3.3. Assume moreover that Ω is C^2, that $(f^n)_n$ is a sequence in $L^2(\Omega)^d$ and that $\nabla y^0 \in L^2(\Omega)^d$.*

a) Suppose $d = 2$. If, for all $n \in \mathbb{N}$, $\nu^{-2}(c_0\alpha^{-1}\sum_{k=0}^{n}\|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu\|y^0\|_2^2)$ is small enough, then there exists at least one solution $y_{n+1} \in H^2(\Omega)^d \cap \boldsymbol{V}$ of (3.3) satisfying

$$\int_\Omega |\nabla y^{n+1}|^2 + \frac{\nu}{4\alpha}\sum_{k=1}^{n+1}\int_\Omega |\Delta y^k|^2 \le \frac{1}{\nu}\Big(\frac{1}{\alpha}\sum_{k=0}^{n}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2\Big). \tag{3.7}$$

b) Suppose $d = 3$. If $\nu^{-2}(\alpha^{-1}\sum_{k=0}^{n-1}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2)$ is small enough, then the conclusion of a) holds true as well.

c) Suppose $d = 2$ and that $(\alpha\nu^5)^{-1}(c_0\alpha^{-1}\sum_{k=0}^{n}\|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu\|y^0\|_2^2)\big(\alpha^{-1}\sum_{k=0}^{n-1}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2\big)$ is small enough. Then, the solution of (3.3) is unique.

In the case $d = 3$, the same conclusion holds true if the quantity $\nu^{-5/2}\alpha^{-1/2}\big(\alpha^{-1}\sum_{k=0}^{n-1}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2\big)$ is small enough.

Proof. From Proposition 2.2, we know that for all $n \in \mathbb{N}^*$, $y^n \in H^2(\Omega)^2 \cap V$. Let now P be the operator of projection from $L^2(\Omega)^d$ into $\boldsymbol{H}$. Taking $w = P\Delta y^{n+1}$ in (3.3) leads to :

(3.8)

$$\alpha\int_\Omega |\nabla y^{n+1}|^2 + \nu\int_\Omega |P\Delta y^{n+1}|^2 = -\int_\Omega f^n P\Delta y^{n+1} + \int_\Omega y^{n+1}.\nabla y^{n+1}.P\Delta y^{n+1} + \alpha\int_\Omega \nabla y^n\nabla y^{n+1}.$$

Recall that

$$\int_\Omega f^n P\Delta y^{n+1} \le \frac{1}{2\nu}\|f^n\|_2^2 + \frac{\nu}{2}\|P\Delta y^{n+1}\|_2^2, \quad \alpha\int_\Omega \nabla y^n\cdot\nabla y^{n+1} \le \frac{\alpha}{2}\int_\Omega |\nabla y^n|^2 + \frac{\alpha}{2}\int_\Omega |\nabla y^{n+1}|^2.$$

a) Assume first that $d = 2$. We can write

$$\Big|\int_\Omega y^{n+1}.\nabla y^{n+1}.P\Delta y^{n+1}\Big| \le \|y^{n+1}\|_\infty\|\nabla y^{n+1}\|_2\|P\Delta y^{n+1}\|_2.$$

We now use (see [19, chapter 5]) that there exist three constants c_1, c_2 and c_3 such that

$$\|\Delta y^{n+1}\|_2 \le c_1\|P\Delta y^{n+1}\|_2, \quad \|y^{n+1}\|_\infty \le c_2\|y^{n+1}\|_2^{\frac{1}{2}}\|\Delta y^{n+1}\|_2^{\frac{1}{2}}$$

and

$$\|\nabla y^{n+1}\|_2 \le c_3\|y^{n+1}\|_2^{\frac{1}{2}}\|\Delta y^{n+1}\|_2^{\frac{1}{2}}.$$

This implies that (for $c = c_1c_2c_3$)

$$\Big| \int_\Omega y^{n+1} \cdot \nabla y^{n+1} \cdot P\Delta y^{n+1} \Big| \leq c\|y^{n+1}\|_2 \|P\Delta y^{n+1}\|_2^2.$$

Recalling (3.8), it results that

$$\frac{\alpha}{2}\int_\Omega |\nabla y^{n+1}|^2 + \left(\frac{\nu}{2} - c\|y^{n+1}\|_2\right)\int_\Omega |P\Delta y^{n+1}|^2 \leq \frac{1}{2\nu}\|f^n\|_2^2 + \frac{\alpha}{2}\int_\Omega |\nabla y^n|^2.$$

We deduce from (3.5), that if $\frac{c_0}{\alpha}\sum_{k=0}^n \|f^k\|^2_{H^{-1}(\Omega)^2} + \nu\|y^0\|_2^2 < c_1\nu^2$ for some constant $c_1 > 0$, then

$$\int_\Omega |\nabla y^{n+1}|^2 + \frac{\nu}{4\alpha}\int_\Omega |P\Delta y^{n+1}|^2 \leq \frac{1}{\nu\alpha}\|f^n\|_2^2 + \int_\Omega |\nabla y^n|^2.$$

Summing then implies (3.7) for all $n \in \mathbb{N}$.

b) We assume now that $d = 3$. Then,

$$\Big| \int_\Omega y_{n+1}.\nabla y_{n+1}.P\Delta y_{n+1} \Big| \leq \|y_{n+1}\|_3 \|\nabla y_{n+1}\|_6 \|P\Delta y_{n+1}\|_2.$$

Again, we use that there exist constants $c_1, c_2 > 0$ such that

$$\|\Delta y_{n+1}\|_2 \leq c_1\|P\Delta y_{n+1}\|_2, \quad \|\nabla y_{n+1}\|_6 \leq c_2\|\Delta y_{n+1}\|_2 \leq c_1c_2\|P\Delta y_{n+1}\|_2$$

so that, for $c = c_1c_2$, we obtain

$$\Big| \int_\Omega y_{n+1}.\nabla y_{n+1}.P\Delta y_{n+1} \Big| \leq c\|y_{n+1}\|_3 \|P\Delta y_{n+1}\|_2^2.$$

Recalling (3.8), it results that

$$\frac{\alpha}{2}\int_\Omega |\nabla y^{n+1}|^2 + \left(\frac{\nu}{2} - c\|y^{n+1}\|_3\right)\int_\Omega |P\Delta y^{n+1}|^2 \leq \frac{1}{2\nu}\|f^n\|_2^2 + \frac{\alpha}{2}\int_\Omega |\nabla y^n|^2. \tag{3.9}$$

Assume that, for all $n \in \mathbb{N}^*$, we have constructed by recurrence an element y^n solution of (3.3) such that

$$\frac{\nu}{4} - c\|y^n\|_3 > 0. \tag{3.10}$$

Then, for all $n \in \mathbb{N}$

$$\int_\Omega |\nabla y^{n+1}|^2 + \frac{\nu}{4\alpha}\int_\Omega |P\Delta y^{n+1}|^2 \leq \frac{1}{\nu\alpha}\|f^n\|_2^2 + \int_\Omega |\nabla y^n|^2 \tag{3.11}$$

and recursively, for all $n \in \mathbb{N}^*$:

$$\int_\Omega |\nabla y^{n+1}|^2 + \frac{\nu}{4\alpha}\sum_{k=1}^{n+1}\int_\Omega |P\Delta y^k|^2 \leq \frac{1}{\nu}\Big(\frac{1}{\alpha}\sum_{k=0}^n \|f^k\|_2^2 + \nu\int_\Omega |\nabla y^0|^2\Big).$$

It remains to construct a sequence $(y^n)_{n\in\mathbb{N}^*}$ solution of (3.3) and satisfying for all $n \in \mathbb{N}^*$ the property (3.10). Let $n \in \mathbb{N}$ fixed. Assume now, that we have constructed, for $k \in \{1, \cdots, n\}$ a solution y^k satisfying (3.3) and $\frac{\nu}{4} - c\|y^k\|_3 > 0$ for $c = c_1c_2$ introduced above. Let $y_1 \in \boldsymbol{V}$ and let $y_2 \in H^2(\Omega)^3 \cap \boldsymbol{V}$ be the unique solution of

$$\alpha\int_\Omega (y_2 - y^n).w + \nu\int_\Omega \nabla y_2.\nabla w + \int_\Omega y_1.\nabla y_2.w = \langle f^n, w\rangle_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} \qquad \forall w \in V.$$

If y_1 satisfies $\frac{\nu}{4c} \geq \|y^1\|_3$, then in view of (3.9),

$$\frac{\alpha}{2}\int_\Omega |\nabla y_2|^2 + \left(\frac{\nu}{2} - c\|y_1\|_3\right)\int_\Omega |P\Delta y_2|^2 \leq \frac{1}{2\nu}\|f^n\|_2^2 + \frac{\alpha}{2}\int_\Omega |\nabla y^n|^2$$

and consequently

$$\frac{\alpha}{2}\int_\Omega |\nabla y_2|^2 + \frac{\nu}{4}\int_\Omega |P\Delta y_2|^2 \leq \frac{1}{2\nu}\|f^n\|_2^2 + \frac{\alpha}{2}\int_\Omega |\nabla y^n|^2.$$

(3.11) then implies

$$\int_\Omega |\nabla y_2|^2 + \frac{\nu}{2\alpha}\int_\Omega |P\Delta y_2|^2 \le \frac{1}{\nu}\Big(\frac{1}{\alpha}\sum_{k=0}^{n}\|f^k\|_2^2 + \nu\int_\Omega |\nabla y^0|^2\Big). \tag{3.12}$$

We now use that there exists a constant $c_1 > 0$ such that, for all $n \in \mathbb{N}$ $\|y_2\|_3 \le c_1\|\nabla y_2\|_2$ to obtain

$$\|y_2\|_3^2 \le \frac{c_1}{\nu}\Big(\frac{1}{\alpha}\sum_{k=0}^{n}\|f^k\|_2^2 + \nu\int_\Omega |\nabla y^0|^2\Big).$$

Consequently, if

$$c_1\Big(\frac{1}{\alpha}\sum_{k=0}^{n}\|f^k\|_2^2 + \nu\int_\Omega |\nabla y^0|^2\Big) \le \frac{\nu^2}{4c}$$

then $\frac{\nu}{4c} \ge \|y_2\|_3$.

We then introduce the application $T : C \to C$, $y_1 \mapsto y_2$ where C is the closed convex set of $\boldsymbol{V}$ defined by $C = \{y \in \boldsymbol{V},\ \frac{\nu}{4c} \ge \|y\|_3\}$. Let us check that T is continuous. Let $y_1 \in C$ et $z_1 \in C$, $y_2 = T(y_1)$ et $z_2 = T(z_1)$ so that

$$\alpha\int_\Omega (z_2-y_2).w + \nu\int_\Omega \nabla(z_2-y_2).\nabla w + \int_\Omega y_1.\nabla(y_2-z_2).w + \int_\Omega (y_1-z_1).\nabla z_2.w = 0 \qquad \forall w \in \boldsymbol{V}.$$

and then, $w = z_2 - y_2$:

$$\begin{aligned}\alpha\int_\Omega |z_2-y_2|^2 + \nu\int_\Omega |\nabla(z_2-y_2)|^2 &\le \Big|\int_\Omega (y_1-z_1).\nabla z_2.(z_2-y_2)\Big| \\ &\le c\|\nabla(y_1-z_1)\|_2\|\nabla z_2\|_2\|z_2-y_2\|_3 \\ &\le c\|\nabla(y_1-z_1)\|_2\end{aligned}$$

using (3.12); this implies the continuity of T. On the other hand, since $y^2 \in H^2(\Omega)^3$, T is relatively compact. The Schauder theorem allows to affirm that T has a fixe point $y \in C$, that is, a solution $y^{n+1} \in C$ of (3.3).

c) Let $n \in \mathbb{N}$ and let $y_1^{n+1}, y_2^{n+1} \in \boldsymbol{V}$ be two solutions of (3.3). Let $Y = y_1^{n+1} - y_2^{n+1}$. Then,

$$\alpha\int_\Omega Y.w + \nu\int_\Omega \nabla Y.\nabla w + \int_\Omega y_2^{n+1}.\nabla Y.w + \int_\Omega Y.\nabla y_1^{n+1}.w = 0 \qquad \forall w \in \boldsymbol{V}$$

and in particular, for $w = Y$ (since $\int_\Omega y_2^{n+1}.\nabla Y.Y = 0$) and $d = 2$

$$\begin{aligned}\alpha\int_\Omega |Y|^2 + \nu\int_\Omega |\nabla Y|^2 = -\int_\Omega Y.\nabla y_1^{n+1}.Y &= \int_\Omega Y.\nabla Y.y_1^{n+1} \le c\|y_1^{n+1}\|_\infty\|\nabla Y\|_2\|Y\|_2 \\ &\le c\|y_1^{n+1}\|_2^{1/2}\|P\Delta y_1^{n+1}\|_2^{1/2}\|\nabla Y\|_2\|Y\|_2 \\ &\le \alpha\|Y\|_2^2 + \frac{c}{\alpha}\|y_1^{n+1}\|_2\|P\Delta y_1^{n+1}\|_2\|\nabla Y\|_2^2\end{aligned}$$

leading to

$$\Big(\nu - \frac{c}{\alpha}\|y_1^{n+1}\|_2\|P\Delta y_1^{n+1}\|_2\Big)\|\nabla Y\|_2^2 \le 0.$$

If

$$\|y_1^{n+1}\|_2\|P\Delta y_1^{n+1}\|_2 < \frac{\nu\alpha}{c}, \tag{3.13}$$

then $Y = 0$ and the solution is unique. But, from (3.5) and (3.7),

$$\|y_1^{n+1}\|_2^2\|P\Delta y_1^{n+1}\|_2^2 \le \frac{4\alpha}{\nu^3}\Big(\frac{c_0}{\alpha}\sum_{k=0}^{n}\|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu\|y^0\|_2^2\Big)\Big(\frac{1}{\alpha}\sum_{k=0}^{n}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2\Big).$$

Therefore, if there exists a constant c_1 such that $(\frac{c_0}{\alpha}\sum_{k=0}^{n}\|f^k\|_{H^{-1}(\Omega)^2}^2 + \nu\|y^0\|_2^2)(\frac{1}{\alpha}\sum_{k=0}^{n}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2) < c_1\nu^5\alpha$, then (3.13) holds true.

If $d = 3$, we have

$$\begin{aligned}\alpha\int_\Omega |Y|^2 + \nu\int_\Omega |\nabla Y|^2 &= -\int_\Omega Y.\nabla y_1^{n+1}.Y = \int_\Omega Y.\nabla Y.y_1^{n+1} \leq c\|y_1^{n+1}\|_\infty \|\nabla Y\|_2\|Y\|_2\\ &\leq c\|\nabla y_1^{n+1}\|_2^{1/2}\|P\Delta y_1^{n+1}\|_2^{1/2}\|\nabla Y\|_2\|Y\|_2\\ &\leq \alpha\|Y\|_2^2 + \frac{c}{\alpha}\|\nabla y_1^{n+1}\|_2\|P\Delta y_1^{n+1}\|_2\|\nabla Y\|_2^2\end{aligned}$$

and therefore $(\nu - \frac{c}{\alpha}\|\nabla y_1^{n+1}\|_2\|P\Delta y_1^{n+1}\|_2)\|\nabla Y\|_2^2 \leq 0$. Moreover, from (3.7),

$$\|\nabla y_1^{n+1}\|_2\|P\Delta y_1^{n+1}\|_2 \leq \frac{2\alpha^{1/2}}{\nu^{3/2}}\Big(\frac{1}{\alpha}\sum_{k=0}^{n}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2\Big).$$

Therefore, if there exists a constant $c > 0$ such that $\frac{1}{\alpha}\sum_{k=0}^{n}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2 < c\nu^{5/2}\alpha^{1/2}$, then, arguing as before, $\|\nabla Y\|_2^2 \leq 0$ and $Y = 0$. □

Proposition 3.2 then allows to obtain the following estimation of $\|y^{n+1} - y^n\|_2$ in term of the parameter α.

Theorem 3.3. *We assume that Ω is C^2, that $(f^n)_n$ is a sequence in $L^2(\Omega)^d$ satisfies $\alpha^{-1}\sum_{k=0}^{+\infty}\|f^k\|_2 < +\infty$, that $\nabla y^0 \in L^2(\Omega)^d$ and that forall $n \in \mathbb{N}$, $y_{n+1} \in H^2(\Omega)^d \cap \boldsymbol{V}$ is a solution of (3.3) satisfying (3.7). Then, the sequence $(y^n)_n$ satisfies*

$$\|y^{n+1} - y^n\|_2 = \mathcal{O}(\alpha^{-1/2}\nu^{-3/4}). \tag{3.14}$$

Proof. For all $n \in \mathbb{N}$, $w = y^{n+1} - y^n$ in (3.3):

$$\alpha\|y^{n+1}-y^n\|_2^2+\nu\|\nabla y^{n+1}\|_2^2 \leq \Big|\int_\Omega y^{n+1}.\nabla y^{n+1}.(y^{n+1}-y^n)\Big|+\Big|\int_\Omega f^n.(y^{n+1}-y^n)\Big|+\nu\Big|\int_\Omega \nabla y^n.\nabla y^{n+1}\Big|.$$

Moreover,

$$\Big|\int_\Omega y^{n+1}.\nabla y^{n+1}.(y^{n+1}-y^n)\Big| \leq c\|\nabla y^{n+1}\|_2^2\|\nabla(y^{n+1}-y^n)\|_2 \leq c\|\nabla y^{n+1}\|_2^2(\|\nabla y^{n+1}\|_2+\|\nabla y^n)\|_2),$$

$$\Big|\int_\Omega f^n.(y^{n+1}-y^n)\Big| \leq \frac{1}{2\alpha}\|f^n\|_2 + \frac{\alpha}{2}\|y^{n+1}-y^n\|_2^2,$$

and

$$\nu\Big|\int_\Omega \nabla y^n.\nabla y^{n+1}\Big| \leq \frac{\nu}{2}\|\nabla y^{n+1}\|_2^2 + \frac{\nu}{2}\|\nabla y^n\|_2^2.$$

Therefore,

$$\alpha\|y^{n+1}-y^n\|_2^2+\nu\|\nabla y^{n+1}\|_2^2 \leq c\|\nabla y^{n+1}\|_2^2(\|\nabla y^{n+1}\|_2+\|\nabla y^n\|_2) + \frac{1}{\alpha}\|f^n\|_2 + \nu\|\nabla y^n\|_2^2.$$

But, from (3.7) we deduce that for all $n \in \mathbb{N}$

$$\int_\Omega |\nabla y^{n+1}|^2 \leq \frac{1}{\nu}\Big(\frac{1}{\alpha}\sum_{k=0}^{+\infty}\|f^k\|_2^2 + \nu\|\nabla y^0\|_2^2\Big) := \frac{C}{\nu}$$

and thus, since $\nu < 1$

$$\alpha\|y^{n+1}-y^n\|_2^2+\nu\|\nabla y^{n+1}\|_2^2 \leq \frac{2cC^{3/2}}{\nu^{3/2}} + 2C \leq \frac{C_1}{\nu^{3/2}}$$

leading to $\|y^{n+1}-y^n\|_2^2 = O(\frac{1}{\alpha\nu^{3/2}})$ as announced. □

Therefore, if the time step discretization $\delta t = 1/\alpha$ is chosen small enough according to the value of the viscosity constant, two consecutive elements of the sequence $\{y^n\}_{n\in\mathbb{N}}$ are close from each other.

Eventually, a similar result may be obtained in the case $\theta \in (0, 1)$.

4. Minimizing sequences for E - Gradient method and Newton

4.1. **Conjugate gradient.** The appropriate tool to produce minimizing sequence for the functional E is gradient method. Among them, the Polak-Ribière version, commonly used, of the conjugate gradient (CG for short in the sequel) algorithm (see [8]) have shown its efficiency in the similar context analyzed in [15, 16, 14]. The CG algorithm reads as follows :

- *Step 0: Initialization* - Given any $\eta > 0$ and any $y_0 \in \boldsymbol{V}$, compute the residual $\overline{y}_0 \in \boldsymbol{V}$ solution of
$$(\overline{y}_0, Y)_{\boldsymbol{V}} = E'(y_0) \cdot Y, \quad \forall Y \in \boldsymbol{V}.$$
If $\|\overline{y}_0\|_{\boldsymbol{V}}/\|y_0\|_{\boldsymbol{V}} \leq \eta$ take $y = y_0$ as an approximation of a minimum of E. Otherwise, set $w_0 = \overline{y}_0$.

  For $k \geq 0$, assuming y_k, $\overline{y}_k$, w_k being known with $\overline{y}_k$ and w_k both different from zero, compute $y_{k+1}, \overline{y}_{k+1}$, and if necessary w_{k+1} as follows:

- *Step 1: Steepest descent* - Set $y_{k+1} = y_k - \lambda_k w_k$ where $\lambda_k \in \mathbb{R}$ is the solution of the one-dimensional minimization problem

$$\text{minimize} \quad E(y_k - \lambda w_k) \quad \text{over}\, \lambda \in \mathbb{R}^+. \tag{4.1}$$

Then, compute the residual $\overline{y}_{k+1} \in \boldsymbol{V}$ from the relation

$$(\overline{y}_{k+1}, w)_{\boldsymbol{V}} = E'(y_{k+1}) \cdot w, \quad \forall w \in \boldsymbol{V}, \tag{4.2}$$

which rewrites as follows :

$$\alpha \int_\Omega \overline{y}_{k+1} \cdot w + \int_\Omega \nabla \overline{y}_{k+1} \cdot \nabla w = -\int_\Omega \alpha v_k \cdot w + \nu \nabla v_k \cdot \nabla w + w \cdot \nabla y_k \cdot v_k + y_k \cdot \nabla w \cdot v_k, \quad \forall w \in \boldsymbol{V}. \tag{4.3}$$

- *Step 2: Convergence testing and construction of the new descent direction* - If $\|\overline{y}_{k+1}\|_{\boldsymbol{V}}/\|\overline{y}_0\|_{\boldsymbol{V}} \leq \eta$ take $y = y_{k+1}$; otherwise compute

$$\gamma_k = \frac{(\overline{y}_{k+1}, \overline{y}_{k+1} - \overline{y}_k)_{\boldsymbol{V}}}{(\overline{y}_k, \overline{y}_k)_{\boldsymbol{V}}}, \qquad w_{k+1} = \overline{y}_{k+1} + \gamma_k w_k. \tag{4.4}$$

Then do $k = k + 1$, and return to step 1.

$\overline{y}_k$ is the gradient associated to $E(y_k)$: it satisfies $E'(y_k) \cdot \overline{y}_k = \|\overline{y}_k\|_V^2$ for all $k > 0$. $\overline{y}_k$ vanishes when E' (and so E) vanishes.

Remark 4.1. *For any real λ and any $y_k, w_k \in \boldsymbol{V}$ we get the following expansion :*

$$\begin{aligned} E(y_k - \lambda w_k) = E(y_k) &- \lambda \int_\Omega (\alpha v_k \overline{v}_k + \nabla v_k \cdot \nabla \overline{v}_k) \\ &+ \frac{\lambda^2}{2} \int_\Omega (\alpha |\overline{v}_k|^2 + |\nabla \overline{v}_k|^2 + 2(\alpha v_k \overline{\overline{v}}_k + \nabla v_k \cdot \nabla \overline{\overline{v}}_k)) \\ &- \lambda^3 \int_\Omega \alpha \overline{v}_k \overline{\overline{v}}_k + \nabla \overline{v}_k \cdot \nabla \overline{\overline{v}}_k + \frac{\lambda^4}{2} \int_\Omega \alpha |\overline{\overline{v}}_k|^2 + |\nabla \overline{\overline{v}}_k|^2 \end{aligned} \tag{4.5}$$

where v_k, $\overline{v}_k \in \boldsymbol{V}$ and $\overline{\overline{v}}_k \in \boldsymbol{V}$ solves respectively

$$\begin{aligned} \alpha \int_\Omega v_k \cdot w + \int_\Omega \nabla v_k \cdot \nabla w + \alpha \int_\Omega y_k \cdot w + \nu \int_\Omega \nabla y_k \cdot \nabla w + \int_\Omega y_k \cdot \nabla y_k \cdot w \\ = < f^n, w >_{H^{-1}(\Omega)^d \times H_0^1(\Omega)^d} + \alpha \int_\Omega g^n \cdot w, \quad \forall w \in \boldsymbol{V}, \end{aligned} \tag{4.6}$$

$$\alpha \int_\Omega \overline{v}_k \cdot w + \int_\Omega \nabla \overline{v}_k \cdot \nabla w + \alpha \int_\Omega w_k \cdot w + \nu \int_\Omega \nabla w_k \cdot \nabla w + \int_\Omega w_k \cdot \nabla y_k \cdot w + y_k \cdot \nabla w_k \cdot w = 0, \quad \forall w \in \boldsymbol{V}, \tag{4.7}$$

and

$$\alpha \int_\Omega \overline{\overline{v}}_k \cdot w + \int_\Omega \nabla \overline{\overline{v}}_k \cdot \nabla w + \int_\Omega w_k \cdot \nabla w_k \cdot w = 0, \quad \forall w \in \boldsymbol{V}. \tag{4.8}$$

□

Consequently, each iteration of the CG algorithm requires the resolution of four Stokes problems, namely (4.3), (4.6), (4.7) and (4.8). The incompressibility condition is taken into account with a Lagrange multiplier. Remark that the matrix (to be invert) associated to those four problems is the same and does not change from an iteration to the next one.

4.2. **Gradient and Newton type method.** Very interestingly, equality (2.18) shows that $-Y_1$ given by the solution of (2.11) is a descent direction for the functional E. Remark also, in view of (2.11), that the corrector V associated to Y_1, given by (2.17) with $Y = Y_1$, is nothing else than the corrector v itself. Therefore, we can define an another minimizing sequence y_k as follows:

$$\begin{cases} y_0 \in \boldsymbol{V}, \\ y_{k+1} = y_k - \lambda_k Y_{1,k}, \quad k > 0, \\ \lambda_k = \operatorname{argmin}_{\lambda \in \mathbb{R}^+} E(y_k - \lambda Y_{1,k}) \end{cases} \tag{4.9}$$

where $Y_{1,k}$ solves the formulation

$$\alpha \int_\Omega Y_{1,k} \cdot w + \nu \int_\Omega \nabla Y_{1,k} \cdot \nabla w + \int_\Omega (y_k \cdot \nabla Y_{1,k} + Y_{1,k} \cdot \nabla y_k) \cdot w = -\alpha \int_\Omega v_k \cdot w - \int_\Omega \nabla v_k \cdot \nabla w, \forall w \in \boldsymbol{V} \tag{4.10}$$

leading (see (2.18)) to $E'(y_k) \cdot Y_{1,k} = 2E(y_k)$. The direction $Y_{1,k}$ vanishes when E vanishes. Remark that the minimization of the real function $\lambda \to E(y_k - \lambda Y_{1,k})$ is easily performed as we check from (4.5) that, for all $\lambda \in \mathbb{R}$,

$$E(y_k - \lambda Y_{1,k}) = (1-\lambda)^2 E(y_k) + \lambda^2 (1-\lambda) \int_\Omega \alpha v_k \overline{\overline{v}}_k + \nabla v_k \nabla \overline{\overline{v}}_k + \frac{\lambda^4}{2} \int_\Omega \alpha |\overline{\overline{v}}_k|^2 + |\nabla \overline{\overline{v}}_k|^2 \tag{4.11}$$

where $v_k \in \boldsymbol{V}$ solves (4.6) and $\overline{\overline{v}}_k \in \boldsymbol{V}$ solves

$$\alpha \int_\Omega \overline{\overline{v}}_k \cdot w + \int_\Omega \nabla \overline{\overline{v}}_k \cdot \nabla w + \int_\Omega Y_{1,k} \cdot \nabla Y_{1,k} \cdot w = 0, \quad \forall w \in \boldsymbol{V}. \tag{4.12}$$

Remark 4.2. *Contrary to the CG algorithm of section 4.1, the matrix (associated to the bilinear form of (4.12)) to be invert varies here with k. To avoid this fact, one may replaced (4.9) by*

$$\begin{cases} y_0 \in \boldsymbol{V}, \\ y_{k+1} = y_k - \lambda_k \widetilde{Y}_{1,k}, \quad k > 0, \\ \lambda_k = \mathit{argmin}_{\lambda \in \mathbb{R}^+} E(y_k - \lambda \widetilde{Y}_{1,k}) \end{cases} \tag{4.13}$$

where $\widetilde{Y}_{1,k}$ solves the formulation

$$\alpha \int_\Omega \widetilde{Y}_{1,k} \cdot w + \nu \int_\Omega \nabla \widetilde{Y}_{1,k} \cdot \nabla w + \int_\Omega (y_0 \cdot \nabla \widetilde{Y}_{1,k} + \widetilde{Y}_{1,k} \cdot \nabla y_0) \cdot w = -\alpha \int_\Omega v_k \cdot w - \int_\Omega \nabla v_k \cdot \nabla w, \forall w \in \boldsymbol{V}. \tag{4.14}$$

It is interesting to note that the sequence $\{y_k\}_{(k>0)}$ obtained from (4.9) if we fixe $\lambda_k = 1$ for all k, i.e. $\{y_0 \in \boldsymbol{V}, y_{k+1} = y_k - Y_{1,k}, k \geq 0\}$, is exactly the sequence associated to the Newton-Raphson method to solve directly the weak formulation (1.5) of (1.3). Such algorithm reads as follows :

$$\begin{cases} y_0 \in \boldsymbol{V}, \\ \partial_y F(y_k, z) \cdot (y_{k+1} - y_k) = -F(y_k, z), \qquad \forall z \in \boldsymbol{V}, \quad \forall k \geq 0, \end{cases} \tag{4.15}$$

and converges to a solution $\overline{y}$ if $\partial_y F(y, z)$ is an isomorphism and Lipschitz-continuous with respect to y in the closed ball containing $\overline{y}$, see [4, Theorem 6.3]. Since both sequences defined

by (4.9) and (4.15) coincides for $\lambda_k = 1$, Algorithm 4.9 can therefore be seen as an improvement of the Newton algorithm with an order of convergence at least equal to 2 (in a neighborhood of a solution). In view of (4.11), they share the same behavior as soon as the term $\|\nabla\overline{\overline{v}}_k\|_2^2$ becomes small. Similarly, the algorithm (4.13) with $\lambda_k = 1$ coincides with the quasi Newton algorithm. More precisely, the following proposition show how the minimization of the parameter λ_k improves the robustness of the Newton method, usually used to solve (1.5).

Proposition 4.3. *Let $(y_k)_{k\in\mathbb{N}}$ be the sequence defined by (4.9) with first element y_0 and let $\overline{y}$ be the solution of the Navier-Stokes equation. Assume that the sequence $\{y_k\}_k$ is in the ball $\mathbb{B}$ defined in Proposition 2.6. Then, the sequence $(\|y_k - \overline{y}\|)_{k\in\mathbb{N}}$ converges to zero at least linearly. Once $\|y_m - \overline{y}\| \le c\,\nu$ for some m, the convergence of the sequence $(\|y_k - \overline{y}\|)_{k>m}$ is quadratic.*

Proof. According to (4.11), we have $E(y_{k+1}) = (1-\lambda_k)^2 E(y_k) + \lambda_k^2(1-\lambda_k)A_k + \lambda_k^4 B_k$ with

$$\begin{cases} A_k = \displaystyle\int_\Omega \alpha v_k\cdot\overline{\overline{v}}_k + \nabla v_k\cdot\nabla\overline{\overline{v}}_k \le \big(\alpha\|v_k\|_2^2 + \|\nabla v_k\|_2^2\big)^{1/2}\big(\alpha\|\overline{\overline{v}}_k\|_2^2 + \|\nabla\overline{\overline{v}}_k\|_2^2\big)^{1/2}, \\ B_k = \dfrac{1}{2}\big(\alpha\|\overline{\overline{v}}_k\|_2^2 + \|\nabla\overline{\overline{v}}_k\|_2^2\big). \end{cases} \tag{4.16}$$

Consequently $|A_k| \le 2\sqrt{E(y_k)}\sqrt{B_k}$ and then

$$\begin{aligned} E(y_{k+1}) &\le (1-\lambda_k)^2 E(y_k) + 2\lambda_k^2|1-\lambda_k|\sqrt{E(y_k)}\sqrt{B_k} + \lambda_k^4 B_k \\ &\le \left(|1-\lambda_k|\sqrt{E(y_k)} + \lambda_k^2\sqrt{B_k}\right)^2. \end{aligned} \tag{4.17}$$

Moreover, from (4.12), if $d = 2$,

$$2B_k \le \|Y_{1,k}\|_2^2\|\nabla Y_{1,k}\|_2^2. \tag{4.18}$$

The bound (2.14) implies that $\int_\Omega |\nabla Y_{1,k}|^2 \le \frac{1}{\nu}\max(1,\frac{2}{\nu})\int_\Omega \alpha|v_k|^2 + |\nabla v_k|^2 = \frac{2}{\nu}\max(1,\frac{2}{\nu})E(y_k)$ so that

$$2B_k \le c_0 c_\nu^2 E(y_k)^2, \quad c_\nu := \frac{2}{\nu}\max(1,\frac{2}{\nu})$$

and then $A_k \le c_0 c_\nu E(y_k)^{3/2}$ where c_0 is the Poincaré constant. Estimate (4.17) then becomes

$$\sqrt{E(y_{k+1})} \le \sqrt{E(y_k)}\left(|1-\lambda_k| + \lambda_k^2\widetilde{c}_\nu\sqrt{E(y_k)}\right), \qquad \widetilde{c}_\nu := \sqrt{c_0/2}c_\nu = \mathcal{O}(\nu^{-2}). \tag{4.19}$$

We introduce the polynomial p as follows :

$$p(\lambda) := \left(|1-\lambda| + \lambda^2\widetilde{c}_\nu\sqrt{E(y_k)}\right)$$

so that $\widetilde{c}_\nu\sqrt{E(y_{k+1})} \le p(\lambda_k)\widetilde{c}_\nu\sqrt{E(y_k)}$.

• Assume that $\widetilde{c}_\nu\sqrt{E(y_k)} < 1$ for some k. Then, writing that $p(\lambda_k) \le p(1) = \widetilde{c}_\nu\sqrt{E(y_k)}$, we obtain that $(\widetilde{c}_\nu\sqrt{E(y_{k+1})}) \le (\widetilde{c}_\nu\sqrt{E(y_k)})^2$. This implies that the sequence $\{\widetilde{c}_\nu\sqrt{E(y_m)}\}_{(m\ge k)}$ decreases to zero with a quadratic rate. In particular, if $\widetilde{c}_\nu\sqrt{E(y_0)} \le 1$ and if we fix $\lambda_k = 1$ for all $k \ge 0$ in (4.9), we recover the order two of convergence of Newton type methods. In view of (2.8), $\widetilde{c}_\nu\sqrt{E(y_0)} \le 1$ this means that $\|y_0 - \overline{y}\|_{\boldsymbol{V}}$ is the order of $c\,\nu$ for some $c > 0$. If $\widetilde{c}_\nu\sqrt{E(y_0)} > 1$, the convergence of the Newton method is not guarantees. On the other hand, the decrease of $\{E(y_k)\}_k$ together with the radius of convergence can be improved if the step λ_k, not necessarily taken equal to one, is chosen at each iterate in order to minimize the value of $p(\lambda_k)$.

• Assume that $\widetilde{c}_\nu\sqrt{E(y_k)} \ge 1$ for some k. In that case, p reaches a unique minimum for $\lambda_k = 1/(2\widetilde{c}_\nu\sqrt{E(y_k)}) \in (0,1/2)$ for which $p(\lambda_k) = 1 - \frac{\lambda_k}{2} \in (0,1)$ leading to (in view of (4.19))

$$\widetilde{c}_\nu\sqrt{E(y_{k+1})} \le= p(\lambda_k)\widetilde{c}_\nu\sqrt{E(y_k)} = \left(1 - \frac{1}{4\widetilde{c}_\nu\sqrt{E(y_k)}}\right)\widetilde{c}_\nu\sqrt{E(y_k)}.$$

This inequality implies that the sequence $\{\widetilde{c}_\nu\sqrt{E(y_k)}\}_{k\in\mathbb{N}}$ strictly decreases and then that the sequence $\{p(\lambda_k)\}_{k\in\mathbb{N}}$ decreases as well. This implies that the sequence $\{\widetilde{c}_\nu\sqrt{E(y_k)}\}_{k\in\mathbb{N}}$ decreases to zero at least linearly. In view of the discussion above, once $\widetilde{c}_\nu\sqrt{E(y_k)}$ is less than one, the decrease is quadratic.

Consequently, the optimization of the parameter λ_k improves significantly the Newton algorithm. Observe from (4.19) that the optimal λ_k goes to 1 as $\sqrt{E(y_k)}$ goes to zero. Our experiments in Section 5 will confirm these properties. □

The convergence of the algorithm (4.9) is ensured as soon as the initial guess y_0 such that y_k stays in the ball $\mathbb{B}$ for all k. In view of the definition of the sequence $\{y_k\}_{k\in\mathbb{N}}$, we can write, for all $k>0$,

$$\|y_{k+1}\|_{\boldsymbol{V}} \leq \|y_0\|_{\boldsymbol{V}} + \sum_{m=0}^{k} |\lambda_m| \|Y_{1,m}\|_{\boldsymbol{V}}.$$

Using that the optimal step λ_m is in $(0,1)$ and that $\|Y_{1,m}\|_{\boldsymbol{V}} \leq \nu^{-1}\sqrt{E(y_m)}$ (see (2.14)), we arrive at

$$\|y_{k+1}\|_{\boldsymbol{V}} \leq \|y_0\|_{\boldsymbol{V}} + \nu^{-1}\sum_{m=0}^{k}\sqrt{E(y_m)}, \quad \forall k\geq 0.$$

But, $\{\sqrt{E(y_m)}\}_{m\in\mathbb{N}}$ is a decreasing sequence and $\sqrt{E(y_m)} \leq p(\lambda_m)\sqrt{E(y_{m-1})} \leq p(\lambda_0)\sqrt{E(y_{m-1})} \leq p(\lambda_0)^m\sqrt{E(y_0)}$ so that $\sum_{m=0}^{k}\sqrt{E(y_m)} \leq \sqrt{E(y_0)}(1-p^{k+1}(\lambda_0))/(1-p(\lambda_0))$ and

$$\|y_{k+1}\|_{\boldsymbol{V}} \leq \|y_0\|_{\boldsymbol{V}} + \frac{1}{1-p(\lambda_0)}\nu^{-1}\sqrt{E(y_0)}, \quad \forall k\geq 0.$$

For $d=2$, $\mathbb{B} = \{y\in\boldsymbol{V}, \|y\|^2_{\boldsymbol{V}} \leq c\alpha\nu\}$ for some $c=c(\Omega)>0$. Consequently, the sequence $\{y_k\}_{k\in\mathbb{N}}$ remains in $\mathbb{B}$ if $\nu^{-1}\sqrt{E(y_0)}$ is of the order of $\alpha\nu$, that is $\sqrt{E(y_0)} \leq d\alpha\nu^{3/2}$ for some $0<d<c$. In view of (2.23), a sufficient condition is given by

$$\|y_0-\overline{y}\|_{\boldsymbol{V}} \leq c\left(\sqrt{\|\overline{y}\|^2_{\boldsymbol{V}} + 4\alpha\nu^{3/2}} - \|\overline{y}\|_{\boldsymbol{V}}\right)$$

for some $c>0$. Remark that $\sqrt{\|\overline{y}\|^2_{\boldsymbol{V}} + 4\alpha\nu^{3/2}} - \|\overline{y}\|_{\boldsymbol{V}} \leq 2\alpha^{1/2}\nu^{3/4}$. We can then complete Proposition 4.4 as follows.

Corollary 4.4. *Let $(y_k)_{k\in\mathbb{N}}$ be the sequence defined by (4.9) with first element y_0 and let $\overline{y}$ be the solution of the Navier-Stokes equation. Assume that $\sqrt{E(y_0)} \leq \mathcal{O}(\alpha\nu^{3/2})$, then the sequence converges to $\overline{y}$ as least linearly. If moreover, $\sqrt{E(y_0)} \leq \mathcal{O}(\nu^2)$, the convergence is quadratic.*

Remark 4.5. *In the scalar case, the optimization of the parameter associated to the Newton-Raphson method is quite straightforward. Let $f:\mathbb{R}\to\mathbb{R}$ be a smooth enough function in the neighborhood of $\alpha\in\mathbb{R}$ such that $f(\alpha)=0$. For instance, the Householder method given in* [9, Section 4.4] *to iteratively approximate α is as follows :*

$$\begin{cases} \alpha^0 \in \mathbb{R}, \\ \alpha^{n+1} = \alpha^n - \lambda_n \dfrac{f(\alpha^n)}{f'(\alpha^n)}, \quad \lambda_n = 1 + \dfrac{f(\alpha^n)f''(\alpha^n)}{2(f'(\alpha^n))^2}. \end{cases} \tag{4.20}$$

This Newton type method has an order of convergence equals to 3 (in a neighborhood of α). The non constant step λ_n minimizes (at the second order) the functional $\lambda \to f\big(\alpha^n - \lambda f(\alpha^n)/f'(\alpha^n)\big)$ and is closed to one if $f(\alpha^n)$ is closed to 0 (i.e. α^n close to α). An analogous optimization in the variational case provided by the formulation F given in (1.5) is achieved through the introduction of the functional E.

5. Numerical illustrations

In the steady case for $\alpha = 0$, we present numerical experiments, including very small values of ν for two 2D examples. The first one is the well-known channel with a backward facing step. The second one concernes a semi-disk geometry. In both case, the velocity of the fluid is imposed on the boundary.

5.1. **Steady case : Two dimensional channel with a backward facing step.** We consider in the steady and unsteady situation the celebrated test problem of a two-dimensional channel with a backward facing step, described for instance in Section 45 of [6] (see also [10]). We use exactly the geometry and boundary conditions from this reference. The geometry is depicted Figure 1. Dirichlet conditions of the Poiseuille type are imposed on the entrant and sortant sides Γ_1 and Γ_2 of the channel: we impose $y = (4(H - x_2)(x_2 - h)/(H - h)^2, 0)$ on Γ_1 and $y = (4(H - h)x_2(H - x_2)/H^2, 0)$ on Γ_2, with $h = 1, H = 3, l = 3$ and $L = 30$. On the remaining part $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$, the fluid flow is imposed to zero. The external force f is zero.

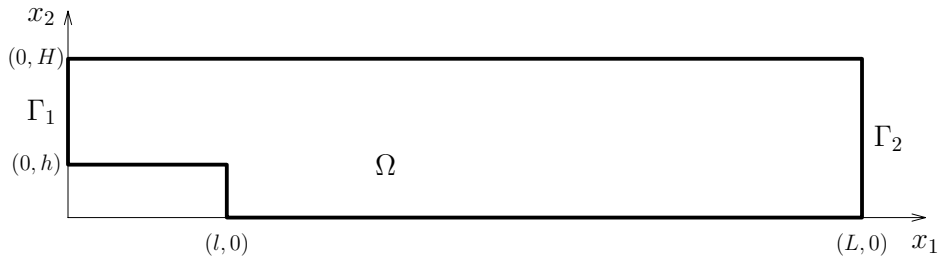


Figure 1. A two-dimensional channel with a step.

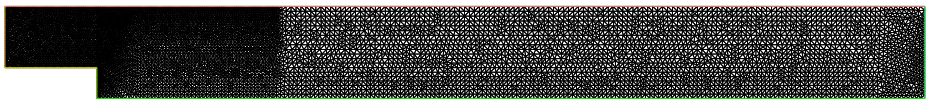


Figure 2. A triangular mesh of the channel - 14 143 triangles and 7 360 vertices.

We consider the extremal problem (2.9) to solve the steady Navier-Stokes equation (1.3). For simplicity, we take here $\alpha = 0$. We compare the gradient algorithms described in the previous section. The first one is the conjugate gradient algorithm coupled with the natural gradient of E. The second one is based on the descent direction Y_1, see (4.9) exhibited in the proof of Proposition 2.6. In both cases, the initial guess is defined as the solution of the corresponding Stokes problem and the scalar extremal problem (4.1) is performed with the Newton-Rasphon method for real function.

The $\mathbb{P}_1/\mathbb{P}_2$ Taylor-Hood finite element, satisfying the Ladyzenskaia-Babushka-Brezzi condition, is employed. We start with a relatively large value of $\nu = 1/150$. Table 1 reports the evolution of the relative quantity $\|y_{k+1} - y_k\|_{\boldsymbol{V}}/\|y_k\|_{\boldsymbol{V}}$ with respect to the iterate k associated to the algorithms (4.9), (4.13), (4.9) with fixed step $\lambda_k = 1$ and conjugate gradient algorithm respectively. A regular mesh composed of 20 868 triangles and 10 792 vertices (similar to the one depicted in Figure 2) is used. Table 2 reports the evolution of the norm of the corrector $\|v_k\|_{\boldsymbol{V}} = \sqrt{2E(y_k)}$, a upper bound of $\|y - y_k\|_{\boldsymbol{V}}$, according to Proposition 2.4. As expected in view of the discussion in Section 4.2, the gradient algorithm (4.9) based on $Y_{1,k}$ is much faster than the CG algorithm based on the natural gradient $\overline{y}_k$. This latter provides however a satisfactory speed of convergence. Moreover, the optimal values for the optimal step λ_k are closed to one, so that the Newton method provides a similar speed of convergence. As the norm of $Y_{1,k}$ goes to zero with k, the last term in (4.11) gets small, and the optimal λ_k gets close to one. Remark as

well that the algorithm (4.13), whose each iterates involves the same matrice to be invert, offers an excellent speed of convergence. In term of CPU times, Algorithm (4.9) and (4.13) require about 53 seconds and 108 seconds respectively and leads to the same approximation. We have notably $\|\nabla \cdot y\|_{L^2(\Omega)} = 1.59 \times 10^{-2}$ and $\|\nabla \cdot y\|_{L^2(\Omega)}/|\Omega| = 1.83 \times 10^{-4}$.

♯ iterate k	(4.9) with $\lambda_k = 1$	(4.9)	(4.13)	CG
1	4.442×10^{-1}	3.798×10^{-1} (0.8545)	3.796×10^{-1}	5.214×10^{-2}
2	1.959×10^{-1}	1.810×10^{-1} (0.9573)	1.592×10^{-1}	4.195×10^{-2}
3	5.609×10^{-2}	4.045×10^{-2} (0.9949)	4.375×10^{-2}	3.276×10^{-2}
4	3.986×10^{-3}	2.223×10^{-3} (1.0006)	6.055×10^{-3}	2.946×10^{-2}
5	2.082×10^{-5}	5.719×10^{-6} (0.9999)	6.808×10^{-3}	2.568×10^{-2}
6	5.912×10^{-10}	4.959×10^{-11} (1)	9.899×10^{-4}	2.290×10^{-2}
7	4.881×10^{-15}	3.299×10^{-15} (1)	9.009×10^{-4}	2.219×10^{-2}
8	–	–	1.486×10^{-4}	2.024×10^{-2}
9	–	–	9.553×10^{-5}	1.952×10^{-2}
10	–	–	2.092×10^{-5}	1.819×10^{-2}
11	–	–	1.396×10^{-5}	1.764×10^{-2}
12	–	–	3.170×10^{-6}	1.723×10^{-2}
13	–	–	1.839×10^{-6}	1.674×10^{-2}
14	–	–	3.809×10^{-7}	1.657×10^{-2}
15	–	–	1.987×10^{-7}	1.606×10^{-2}
26	–	–	4.321×10^{-13}	1.120×10^{-2}
50	–	–	–	3.325×10^{-3}
100	–	–	–	1.756×10^{-3}
200	–	–	–	2.091×10^{-5}

TABLE 1. $\nu = 1/150$; Evolution of $\|y_{k+1} - y_k\|_V / \|y_k\|_{\boldsymbol{V}}$ with respect to k.

For smaller values of ν, the results are qualitatively different. Table 3 reports some norms with respect to k for $\nu = 1/700$. We observe, from the last column, that the Newton method for which λ_k is fixed to one does not converge anymore. Actually, Newton's method, when initialized with the solution of the corresponding Stokes problem, diverges for $\nu \leq 1/250$. On the other hand, the optimization of the step λ_k produces a very fast convergence of the sequence $\{y_k\}_{(k>0)}$. Observe here that the values for the optimal λ_k are not closed to one, during the first iterates. We obtain notably $\|\nabla \cdot y\|_{L^2(\Omega)}/|\Omega| = 5.78 \times 10^{-2}$. In agreement with Proposition 4.4, we also clearly observe from Table 3 that the decrease of $\sqrt{E(y_k)}$ to zero is first linear and then becomes quadratic.

The algorithm (4.13) is a bit more robust than the Newton one as it converges for all ν satisfying $\nu \geq 1/290$ approximately. Finally, as discussed in [11], the CG algorithm converges and produces similar numerical values: the convergence is however slower since about 350 iterates are needed to achieve $\sqrt{2E(y_k)}$ of the order 10^{-3}.

The algorithm (4.9) requires however the initial guess to be close enough to the solution. Initialized with the solution of the corresponding Stokes problem, it diverges for $\nu \leq 1/720$. A continuation method with respect to ν is then necessary in that case. Algorithm 4.9 is also robust with respect to the mesh size: with a twice finer mesh composed of 84 707 triangles and 43 069 vertices, the convergence $\|y^{k+1} - y^k\| \leq 10^{-12}\|y^k\|$ is observed after $k = 18$ iterates (instead of 14 for the previous coarser mesh) leading notably $\|\nabla \cdot y\|_{L^2(\Omega)}/|\Omega| = 3.91 \times 10^{-2}$. Figure 3 depicts the streamlines of the convergent sequence y_k in the cases $\nu = 1/150$ and $\nu = 1/700$. The method allows to capture the shear layer developing in the flow behind the re-entrant corner.

♯ iterate k	(4.9) with $\lambda_k = 1$	(4.9)	(4.13)	CG
1	5.467×10^{-2}	5.467×10^{-2}	5.476×10^{-2}	5.467×10^{-2}
2	2.398×10^{-2}	2.224×10^{-2}	2.222×10^{-2}	3.701×10^{-2}
3	4.953×10^{-3}	4.601×10^{-3}	5.457×10^{-3}	2.917×10^{-2}
4	3.201×10^{-4}	1.565×10^{-4}	9.322×10^{-4}	2.492×10^{-2}
5	1.530×10^{-6}	5.437×10^{-7}	5.191×10^{-4}	2.201×10^{-2}
6	3.650×10^{-11}	4.227×10^{-12}	1.712×10^{-4}	1.995×10^{-2}
7	6.541×10^{-16}	2.541×10^{-16}	1.712×10^{-4}	1.840×10^{-2}
8	–	–	7.852×10^{-5}	1.709×10^{-2}
9	–	–	2.472×10^{-5}	1.603×10^{-2}
10	–	–	8.953×10^{-6}	1.511×10^{-2}
11	–	–	3.424×10^{-6}	1.433×10^{-2}
12	–	–	1.205×10^{-6}	1.363×10^{-2}
13	–	–	4.251×10^{-7}	1.301×10^{-2}
14	–	–	1.366×10^{-7}	1.242×10^{-2}
15	–	–	4.478×10^{-8}	1.187×10^{-2}
26	–	–	1.599×10^{-14}	6.259×10^{-3}
50	–	–	–	2.673×10^{-3}
100	–	–	–	7.583×10^{-4}
200	–	–	–	1.551×10^{-5}

TABLE 2. $\nu = 1/150$; Evolution of $\|v_k\|_{\boldsymbol{V}} = \sqrt{2E(y_k)}$ with respect to k.

♯ iterate k	$\|y_{k+1} - y_k\|_V / \|y_k\|_V$	$\sqrt{2E(y_k)}$	λ_k	$\sqrt{2E(y_k)}$ with $\lambda_k = 1$
1	7.153×10^{-1}	5.467×10^{-2}	0.727	5.467×10^{-2}
2	1.424×10^{-4}	2.791×10^{-2}	4.77×10^{-5}	3.452×10^{-2}
3	2.073×10^{-1}	2.791×10^{-2}	2.01×10^{-2}	8.089×10^{-2}
4	3.538×10^{-1}	2.737×10^{-2}	0.958	5.344×10^{-2}
5	9.138×10^{-2}	7.270×10^{-3}	4.81×10^{-6}	2.409
6	6.244×10^{-2}	2.622×10^{-3}	1.73×10^{-3}	6.115×10^{-1}
7	2.028×10^{-2}	1.078×10^{-3}	0.358	3.944
8	3.695×10^{-3}	2.610×10^{-4}	0.521	9.851×10^{1}
9	7.522×10^{-4}	4.184×10^{-5}	1.098	8.186×10^{1}
10	9.886×10^{-6}	6.014×10^{-7}	0.963	4.385×10^{4}
11	3.872×10^{-6}	1.692×10^{-7}	1.032	1.093×10^{4}
12	6.820×10^{-11}	4.404×10^{-12}	0.9983	3.169×10^{4}
13	1.288×10^{-10}	2.880×10^{-12}	0.9999	1.576×10^{5}
14	6.879×10^{-15}	3.263×10^{-16}	1.	4.068×10^{4}

TABLE 3. $\nu = 1/700$; Results for the algorithm (4.9).

5.2. **Steady case: 2D semi-circular cavity.** We now consider the 2D test discussed in [5]. The geometry is a semi-disk $\Omega = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 < 1/4, x_2 \leq 0\}$ depicted on Figure 4. The velocity is imposed to $y = (g, 0)$ on $\Gamma_0 = \{(x_1, 0) \in \mathbb{R}^2, |x_1| < 1/2\}$ with g vanishing at $x_1 = \pm 1/2$ and close to one elsewhere: we take $g(x_1) = (1 - e^{100(x_1 - 1/2)})(1 - e^{-100(x_1 + 1/2)})$. On the rest $\Gamma_1 = \{(x_1, x_2) \in \mathbb{R}^2, x_2 < 0, x_1^2 + x_2^2 = 1/4\}$ of the boundary the velocity is fixed to zero.

For a regular triangular mesh, composed of 79 628 triangles and 40 205 vertices, leading to a mesh size $h = 6.23 \times 10^{-3}$, the Newton method ($\lambda_k = 1$) initialized with the corresponding Stokes solution, converges up to $\nu^{-1} \approx 500$. On the other hand, the algorithm (4.9) using the

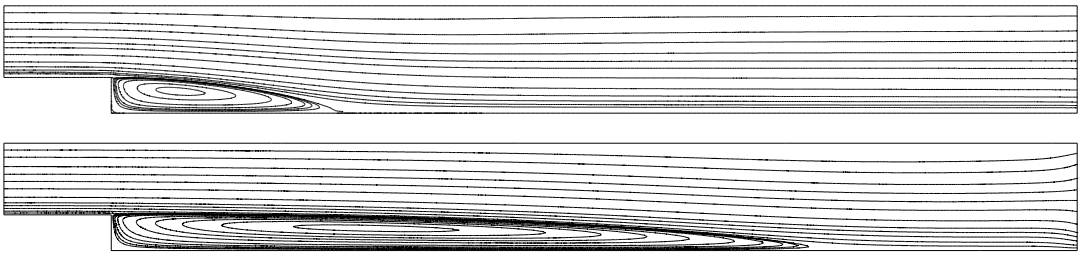


FIGURE 3. Streamlines of the steady state solution at $\nu = 1/150$ (top) and $\nu = 1/700$ (bottom); $L = 30$.

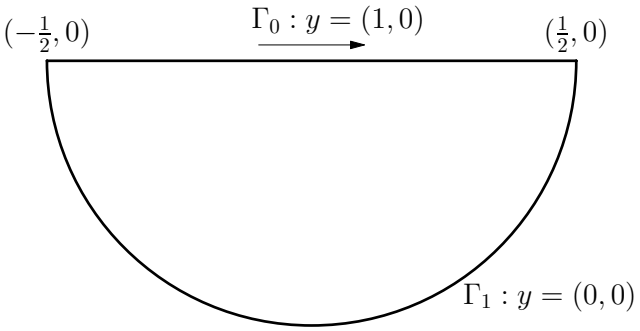


FIGURE 4. Semi-disk geometry.

optimal λ_k still converges up to $\nu^{-1} \approx 910$. Figures 5 depitcs the streamlines of steady state solution corresponding to $\nu^{-1} = 500$ and to $\nu^{-1} = i \times 10^3$ for $i = 1, \cdots, 9$. The figures are in very good agreements with those depicted in [5]. The solution corresponding to $\nu^{-1} = 500$ is obtained from the sequence given (4.9) initialized with the Stokes solution. Seven iterates are necessary to achieve $\sqrt{2E(y)} \approx 3.4 \times 10^{-17}$. The stopping criteria is $\|y_{k+1} - y_k\|_{\boldsymbol{V}} \leq 10^{-12}\|y_k\|_{\boldsymbol{V}}$. Then, the other solutions are obtained by a continuation method with respect to ν taking $\delta\nu^{-1} = 500$. For instance, the solution corresponding to $\nu^{-1} = 5000$ is obtained from the algorithm (4.9) initialized with the steady solution corresponding to $\nu^{-1} = 4500$. Table 4 reports the history of the continuation method and highlights the efficiency of the algorithm (4.9): up to $\nu^{-1} = 9500$, few iterations achieve the convergence of the minimizing sequence $\{y_k\}_{k\in\mathbb{N}}$. From $\nu^{-1} = 10^4$, with a finer mesh (for which the mesh size is $h = 4.37 \times 10^{-3}$), $\delta\nu$ is reduced to $\delta\nu^{-1} = 100$ and leads to convergence beyond $\nu^{-1} = 15000$. Table also reports the minimal value of the streamline function ψ which compare very well with those of [5].

Eventually, we emphasize that the choice of error functional E considered in [2] (see remark 2.9) leads to similar results in term of robustness and convergence, in particular for small values of ν.

5.3. **Unsteady case.** We did not discussed here numerical simulations for $\alpha > 0$ and the implicit scheme (1.2). We refer to [12] where numerical results are compared with those obtained from a fully time-space least-squares approach. We only emphasize that when α is strictly positive, and are fiortiori α large for a small discretization step δt, the algorithm (4.9) remains robust and efficient. This is due to the fact that the α term in the cost E defined in (2.6) increases its coercivity. Moreover, the size of the ball $\mathbb{B}$ of Proposition 2.6, appearing in the discussion of the convergence of the sequence $\{y_k\}_{k\in\mathbb{N}}$ (see also Corollary 4.4) increases with α. Consequently, choosing a large α allows to ensure the convergence of the method and the determination of y^{n+1} from y^n.

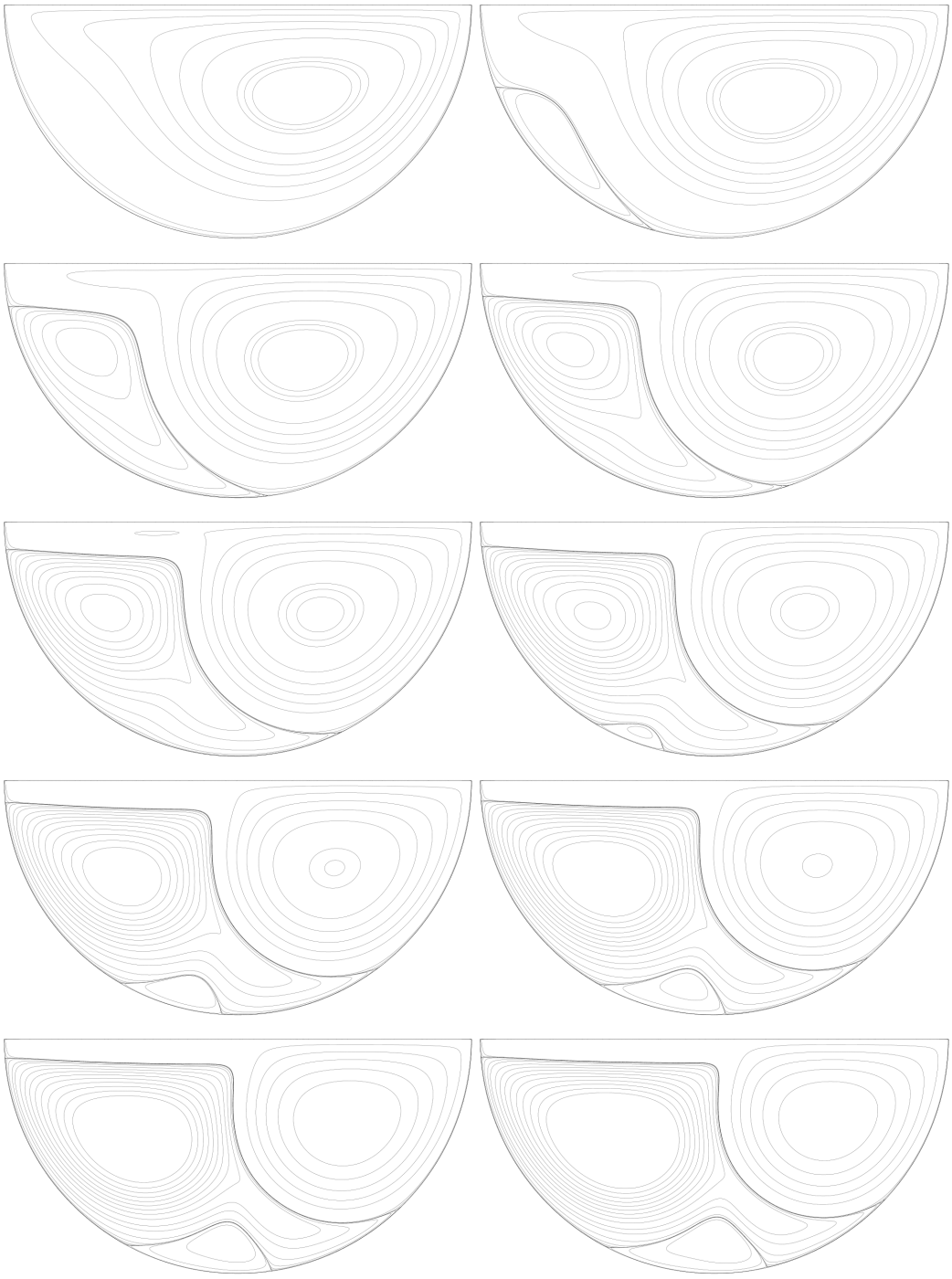


FIGURE 5. Streamlines of the steady state solution for $\nu^{-1} = 500, 1000, 2000, 3000, 4000, 5000, 6000, 7000, 8000$ and $\nu^{-1} = 9000$.

6. Conclusions and perspectives

We have analyzed the so-called H^{-1}-least-squares method introduced forty years ago in [2] allowing to solve the steady nonlinear Navier-Stokes system, in the incompressible regime. We show that any minimizing sequence starting closed enough to a solution converges strongly in $\boldsymbol{V}$. Moreover, the analysis make appear a descent direction for the error functional, different from the

ν^{-1}	♯ it.	$\|\pi\|_{L^2(\Omega)}$	$\|y_1\|_{H_0^1(\Omega)}$	$\|y_2\|_{H_0^1(\Omega)}$	$min_\Omega \psi$	$min_\Omega \psi$ [5]
Stokes $\to$ 500	7	4.31×10^{-2}	4.462	2.489	-0.0766784	–
$500 \to 1000$	7	4.07×10^{-2}	4.919	2.883	-0.0780642	-0.0779
$1000 \to 1500$	6	0.0399	5.2966	3.15371	-0.0775772	–
$1500 \to 2000$	6	0.0393087	5.61222	3.36132	-0.0766604	-0.0763
$2000 \to 2500$	5	0.0388207	5.8849	3.53141	-0.0756008	–
$2500 \to 3000$	5	0.0383734	6.12689	3.67571	-0.074476	-0.0742
$3000 \to 3500$	5	0.0379483	6.34559	3.80097	-0.0733293	–
$3500 \to 4000$	5	0.0375405	6.54581	3.91156	-0.0721912	–
$4000 \to 4500$	6	0.0371478	6.73091	4.01047	-0.0710789	–
$4500 \to 5000$	6	0.0367688	6.90338	4.09986	-0.0699992	-0.0700
$5000 \to 5500$	6	0.0364024	7.06514	4.18132	-0.0689569	–
$5500 \to 6000$	6	0.0360479	7.21768	4.25607	-0.0679526	–
$6000 \to 6500$	6	0.0357043	7.3622	4.32506	-0.0669879	–
$6500 \to 7000$	6	0.0353712	7.49966	4.38908	-0.0660596	–
$7000 \to 7500$	5	0.0350479	7.63085	4.44875	-0.0651669	–
$7500 \to 8000$	5	0.0347341	7.75643	4.5046	-0.0643104	–
$8000 \to 8500$	6	0.0344295	7.87694	4.55705	-0.0634864	–
$8500 \to 9000$	6	0.0341339	7.99287	4.60649	-0.0626943	–
$9000 \to 9500$	11	0.0338468	8.10461	4.65324	-0.0619334	–
$9500 \to 10000$	39	0.0335673	8.21355	4.6937	-0.061204	–
$10000 \to 10100$	5	0.0335126	8.23472	4.70226	-0.061061	–
$10100 \to 10200$	5	0.0334581	8.25574	4.71073	-0.0609195	–
$10200 \to 10300$	5	0.033404	8.27663	4.71911	-0.0607786	–
$10300 \to 10400$	5	0.0333502	8.29738	4.72741	-0.0606386	–
$10400 \to 10500$	5	0.0332967	8.318	4.73563	-0.0605006	–
$10500 \to 10600$	5	0.0332436	8.33848	4.74376	-0.0603624	–

TABLE 4. Continuation method with respect to ν for the solution of steady Navier-Stokes associated to the semi-disk.

one related to the gradient, leading to a second order of convergence, in a neighborhood of any solution. This minimizing sequence actually generalizes the sequence associated to the Newton method for the variational weak formulation of the Navier-Stokes system. Numerical experiments highlight the robustness of the method, even for small values of the viscosity coefficient. The least-squares method can then be used to solve, step by step, standard implicit and unconditionally stable scheme. The value of the time discretization parameter is adjusted according to the value of the viscosity constant, in order to ensure the convergence toward y^{n+1}, approximation of the Navier-Stokes at time t_{n+1}, of the above mentioned sequence, with y^n as a first element. We also emphasize that the least-squares approaches, employed here to treat the Navier-Stokes nonlinearity, can be used to solve another nonlinear equations, as formally done in [16] for a sublinear heat equation. Eventually, we may solve the unsteady Navier-Stokes system by a fully $L^2(0, T, H^{-1}(\Omega))$ least-squares approach. The underlying corrector solves an unsteady Stokes equation; we refer to [12].

REFERENCES

1. Pavel B. Bochev and Max D. Gunzburger, *Least-squares finite element methods*, Applied Mathematical Sciences, vol. 166, Springer, New York, 2009. MR 2490235

2. M. O. Bristeau, O. Pironneau, R. Glowinski, J. Periaux, and P. Perrier, *On the numerical solution of nonlinear problems in fluid dynamics by least squares and finite element methods. I. Least square formulations and conjugate gradie*, Comput. Methods Appl. Mech. Engrg. **17/18** (1979), no. part 3, 619–657. MR 529979
3. R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. (9) **72** (1993), no. 3, 247–286. MR 1225511
4. Vivette Girault and Pierre-Arnaud Raviart, *Finite element methods for Navier-Stokes equations*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986, Theory and algorithms. MR 851383
5. R. Glowinski, G. Guidoboni, and T.-W. Pan, *Wall-driven incompressible viscous flow in a two-dimensional semi-circular cavity*, J. Comput. Phys. **216** (2006), no. 1, 76–91. MR 2223437
6. Roland Glowinski, *Finite element methods for incompressible viscous flow*, Numerical Methods for Fluids (Part 3), Handbook of Numerical Analysis, vol. 9, Elsevier, 2003, pp. 3 – 1176.
7. Roland Glowinski, *Variational methods for the numerical solution of nonlinear elliptic problems*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 86, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2015. MR 3450066
8. William W. Hager and Hongchao Zhang, *A survey of nonlinear conjugate gradient methods*, Pac. J. Optim. **2** (2006), no. 1, 35–58. MR 2548208
9. A. S. Householder, *The numerical treatment of a single nonlinear equation*, McGraw-Hill Book Co., New York-Düsseldorf-London, 1970, International Series in Pure and Applied Mathematics. MR 0388759
10. Bo-Nan Jiang and Louis A. Povinelli, *Least-squares finite element method for fluid dynamics*, Comput. Methods Appl. Mech. Engrg. **81** (1990), no. 1, 13–37. MR 1071091
11. J. Lemoine, A. Munch, and P. Pedregal, *Analysis of continuous H^{-1}-least-squares approaches for the steady Navier-Stokes system*, To appear in Applied Mathematics and Optimization.
12. Jérome Lemoine and Arnaud Münch, *A continuous least-squares method for the unsteady Navier-Stokes system: analysis and applications*, In preparation.
13. J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969. MR 0259693
14. Arnaud Münch, *A least-squares formulation for the approximation of controls for the Stokes system*, Math. Control Signals Systems **27** (2015), no. 1, 49–75. MR 3306625
15. Arnaud Münch and Pablo Pedregal, *A least-squares formulation for the approximation of null controls for the Stokes system*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 13-14, 545–550. MR 3095103
16. ———, *Numerical null controllability of the heat equation through a least squares and variational approach*, European J. Appl. Math. **25** (2014), no. 3, 277–306. MR 3197994
17. Alfio Quarteroni and Alberto Valli, *Numerical approximation of partial differential equations*, Springer Series in Computational Mathematics, vol. 23, Springer-Verlag, Berlin, 1994. MR 1299729
18. Antony Smith and David Silvester, *Implicit algorithms and their linearization for the transient incompressible Navier-Stokes equations*, IMA J. Numer. Anal. **17** (1997), no. 4, 527–545. MR 1476337
19. Luc Tartar, *An introduction to Navier-Stokes equation and oceanography*, Lecture Notes of the Unione Matematica Italiana, vol. 1, Springer-Verlag, Berlin; UMI, Bologna, 2006. MR 2258988
20. Roger Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition. MR 1846644
21. F. Tone and D. Wirosoetisno, *On the long-time stability of the implicit Euler scheme for the two-dimensional Navier-Stokes equations*, SIAM J. Numer. Anal. **44** (2006), no. 1, 29–40. MR 2217369