On superperfection of edge-intersection graphs of paths

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Abstract

The routing and spectrum assignment problem in flexgrid elastic optical networks can be modeled in two phases: a selection of paths in the network and an interval coloring problem in the edge intersection graph of these paths. The interval chromatic number equals the smallest size of a spectrum such that a proper interval coloring is possible, the weighted clique number is a natural lower bound. Graphs where both parameters coincide for all possible non-negative integral weights are called superperfect. We examine the question which minimal non-superperfect graphs can occur in the edge intersection graphs of paths in different underlying networks. We show that for any possible network (even if it is restricted to a path) the resulting edge intersection graphs are not necessarily superperfect and discuss some consequences.

1 Introduction

Flexgrid elastic optical networks constitute a new generation of optical networks in response to the sustained growth of data traffic volumes and demands in communication networks. In such networks, light is used as communication medium between sender and receiver nodes, and the frequency spectrum of an optical fiber is divided into narrow frequency slots of fixed spectrum width. Any sequence of consecutive slots can form a channel that can be switched in the network to create a lightpath (i.e., an optical connection represented by a route and a channel). The *routing and spectrum assignment (RSA) problem* consists of establishing the lightpaths for a set of end-to-end traffic demands, that is, finding a route and assigning an interval of consecutive frequency slots for each demand such that the intervals of lightpaths using a same edge in the network are disjoint, see e.g. [17]. Thereby, the following constraints need to be respected when dealing with the RSA problem:

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1. *spectrum continuity*: the frequency slots remain the same on all the links of a route;
2. *spectrum contiguity*: the frequency slots allocated to a demand must be contiguous;
3. *non-overlapping spectrum*: a frequency slot can be allocated to at most one demand.

The RSA problem is a generalization of the well-studied *routing and wavelength assignment (RWA) problem* that is associated with a fixed grid of frequencies [6]. The former problem has started to receive a lot of attention over the last few years. It has been shown to be NP-hard [3, 18]. In fact, if for each demand the route is already known, the RSA problem reduces to the so-called *spectrum assignment (SA) problem* and only consists of determining the demands' channels. The SA problem has been shown to be NP-hard on paths [16] which makes the SA problem (and thus also the RSA problem) much harder than the RWA problem which is well-known to be polynomially solvable on paths, see e.g. [6].

More formally, for the RSA problem, we are given a network G and a set $\mathcal{D}$ of end-to-end traffic demands where each demand is specified by a pair u, v of distinct nodes in G and the number d_{uv} of required frequency slots. The routing part of the RSA problem consists of selecting a route through G from u to v, i.e. a (u, v)-path P_{uv} in G, for each such traffic demand. The spectrum assignment can then be interpreted as an *interval coloring* of the *edge intersection graph* $I(\mathcal{P})$ of the set $\mathcal{P}$ of selected paths:

- Each path $P_{uv} \in \mathcal{P}$ becomes a node of $I(\mathcal{P})$ and two nodes are joined by an edge if the corresponding paths in G are in conflict as they share an edge (notice that we do not care whether they share nodes).
- Any interval coloring in this graph $I(\mathcal{P})$ weighted with the demands d_{uv} correctly solves the spectrum assignment: we assign a frequency interval of d_{uv} consecutive frequency slots (*spectrum contiguity*) to every node of $I(\mathcal{P})$ (and, thus, to every path $P_{uv} \in \mathcal{P}$ (*spectrum continuity*)) in such a way that the intervals of adjacent nodes are disjoint (*non-overlapping spectrum*).

Let $d \in \mathbb{Z}_+^{|\mathcal{D}|}$ be the vector whose entries d_{uv} are the slot requirements associated with the demands between pairs u, v of nodes in $\mathcal{D}$. The *interval chromatic number* $\chi_I(I(\mathcal{P}), d)$ is the minimum spectrum width such that $I(\mathcal{P})$ weighted with the vector d of traffic demands d_{uv} for each path P_{uv} has a proper interval coloring. Given G and $\mathcal{D}$, the minimum spectrum width of any solution of the RSA problem, thus, equals

$$\chi_I(G, \mathcal{D}) = \min\{\chi_I(I(\mathcal{P}), d) : \mathcal{P} \text{ possible routing of demands } \mathcal{D} \text{ in } G\}.$$

For each routing $\mathcal{P}$, the *weighted clique number* $\omega(I(\mathcal{P}), d)$, also taking the traffic demands d_{uv} as weights, is a natural lower bound for $\chi_I(I(\mathcal{P}), d)$. However, it is not always possible to find a solution with this lower bound as spectrum width, as weighted clique number and interval chromatic number are not always equal.

Graphs where weighted clique number and interval chromatic number coincide for all possible non-negative integral weights are called *superperfect*.

A graph is *perfect* if and only if this holds for every $(0,1)$-weighting d of its nodes, thus every superperfect graph is perfect. A graph $G = (V, E)$ is *comparability* if and only if there exists a partial order $\mathcal{O}$ on $V \times V$ such that $uv \in E$ if and only if u and v are comparable w.r.t. $\mathcal{O}$. Comparability graphs form a subclass of superperfect graphs [12], but there are also superperfect non-comparability graphs such as e.g. even antiholes [8].

A complete list of minimal non-comparability graphs is presented in [7], that are

- odd holes C_{2k+1} for $k \geq 2$ and antiholes $\overline{C}_n$ for $n \geq 6$,
- the graphs J_k and J'_k for $k \geq 2$ and the graphs J''_k for $k \geq 3$ (see Fig. 1),
- the complements of D_k for $k \geq 2$ and of E_k, F_k for $k \geq 1$ (see Fig. 2),
- the complements of $A_1, \ldots, A_{10}$ (see Fig. 3).

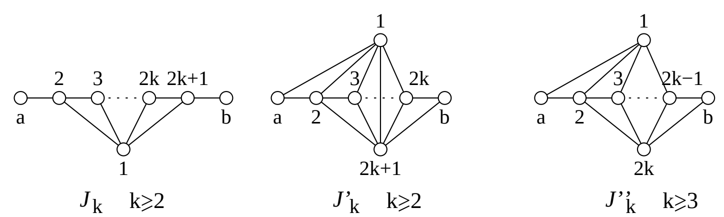

Figure 1: Minimal non-comparability graphs: J_k, J'_k for $k \geq 2$ and J''_k for $k \geq 3$.

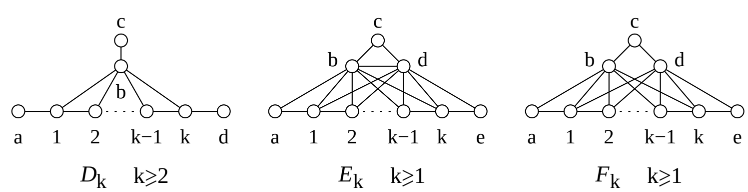

Figure 2: Minimal non-comparability graphs: the complements of D_k, E_k, F_k.

The question which minimal non-comparability graphs are superperfect has been addressed in [1], that are

- even antiholes $\overline{C}_{2k}$ for $k \geq 3$,
- the graphs J''_k for $k \geq 3$,
- the complements of $A_3, \ldots, A_{10}$.

Note that Andreae wrongly determined $\overline{A}_2$ as superperfect which is, in fact, not the case (see Fig. 4 for a weight vector d and an optimal interval coloring showing that $\omega(\overline{A}_2, d) = 5 < 6 = \chi_I(\overline{A}_2, d)$ holds).

The minimal non-comparability graphs which are not superperfect are thus minimal non-superperfect: the graphs $\overline{A}_1$, $\overline{A}_2$ and all

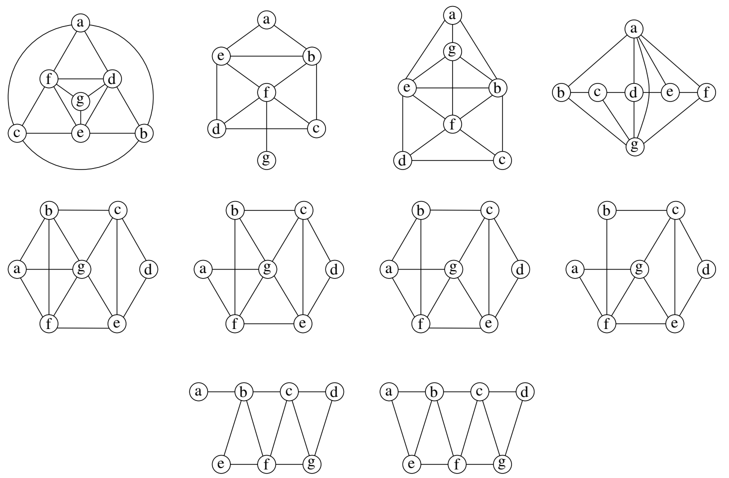

Figure 3: Minimal non-comparability graphs: the graphs $\overline{A_1}, \dots, \overline{A_{10}}$.

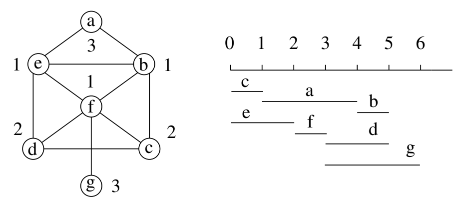

Figure 4: The graph $\overline{A}_2$ together with node weights d and an optimal interval coloring showing $\omega(\overline{A}_2, d) = 5 < 6 = \chi_I(\overline{A}_2, d)$.

- odd holes C_{2k+1} and odd antiholes $\overline{C}_{2k+1}$ for $k \geq 2$,
- the graphs J_k and J'_k for $k \geq 2$,
- the complements of D_k for $k \geq 2$ and of E_k, F_k for $k \geq 1$.

Note that we have $\omega(G, \mathbf{1}) < \chi_I(G, \mathbf{1})$ with $\mathbf{1} = (1, \dots, 1)$ if G is an odd hole or an odd antihole (as they are not perfect), whereas the other minimal non-superperfect graphs from the list are perfect and, thus, $\omega(G, d) < \chi_I(G, d)$ is attained for some $d \neq \mathbf{1}$ (see Fig. 4).

We examine, for different underlying networks G, the question whether or not there is a solution of the RSA problem with

$$\omega(G, \mathcal{D}) = \min\{\omega(I(\mathcal{P}), d) : \mathcal{P} \text{ possible routing of demands } \mathcal{D} \text{ in } G\}$$

as spectrum width which depends on the occurrence of (minimal) non-superperfect graphs in the edge intersection graphs $I(\mathcal{P})$.

Note that for some networks G, the edge intersection graphs form well-studied graph classes: if G is a path (resp. tree, resp. cycle), then $I(\mathcal{P})$ is an *interval graph* (resp. *EPT graph*, resp. *circular-arc graph*). However, if G is a sufficiently large grid, then it is known by [10] that $I(\mathcal{P})$ can be *any* graph. Modern optical networks do not fall in any of these classes, but are 2-connected, sparse planar graphs with small maximum degree with a grid-like structure.

We first study the cases when the underlying network G is a tree or a cycle (see Section 2 and 3). We recall results on EPT graphs and circular-arc graphs from [9, 5] and then discuss which minimal non-comparability non-superperfect graphs can occur. In addition, we exhibit new examples of minimal non-superperfect graphs within these classes.

All of these non-superperfect graphs are inherited for the case when G is an optical network, and we give also representations as edge intersection graphs for the remaining minimal non-comparability non-superperfect graphs. In view of the result on edge intersection graphs of paths in a sufficiently large grid [10], we expect that any further minimal non-superperfect graph has such a representation and give some further new examples of such graphs.

To find new examples, we make use of the complete list of minimal non-comparability graphs found by [7] and the fact that any candidate for a new minimal non-superperfect graph can neither be imperfect nor a comparability graph. Thus, among the graphs with n nodes, the candidates of new minimal non-superperfect graphs are all graphs that are

- perfect (i.e. do not contain odd holes or odd antiholes),
- do not contain any minimal non-superperfect graph with $\leq n$ nodes,
- contain a minimal non-comparability superperfect graph with $< n$ nodes.

Finally, we propose to extend the concept of χ-binding functions introduced in [11] for usual coloring to interval coloring in weighted graphs to describe how large the gap between weighted clique number and interval chromatic number can be in the worst case.

We close with some concluding remarks and open problems.

2 If the network is a tree

If the underlying communication network G is a tree, then there exists exactly one (u, v)-path P_{uv} in G for every traffic demand between a pair u, v of nodes. Hence, if G is a tree, then $\mathcal{P}$ and $I(\mathcal{P})$ are uniquely determined for any set $\mathcal{D}$ of end-to-end traffic demands, and the RSA problem reduces to the spectrum assignment part. The resulting edge intersection graph $I(\mathcal{P})$ belongs to the class of EPT graphs studied in [9].

We recall results from [9] on holes in EPT graphs and examine which minimal non-superperfect graphs can occur in such graphs.

It is known from [9] that EPT graphs are not necessarily perfect as they can contain odd holes. More precisely, Golumbic and Jamison showed the following:

Theorem 1 (Golumbic and Jamison [9]) *If the edge intersection graph $I(\mathcal{P})$ of a collection $\mathcal{P}$ of paths in a tree T contains a hole C_k with $k \geq 4$, then*

T contains a star $K_{1,k}$ with nodes $b, a_1, \ldots, a_k$ and there are k paths $P_1, \ldots, P_k$ in $\mathcal{P}$ such that P_i precisely contains the edges ba_i and ba_{i+1} of this star (where indices are taken modulo k).

Figure 5 illustrates the case of $C_5 = I(\mathcal{P})$.

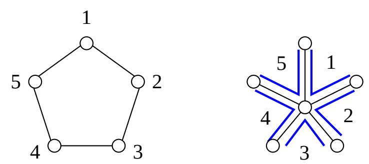

Figure 5: The odd hole $C_5 = I(\mathcal{P})$ with $\mathcal{P}$ in a star.

From the above result, Golumbic and Jamison deduced the possible adjacencies of a hole which further implies that several graphs cannot occur as induced subgraphs of EPT graphs, including the complement of the P_6 and the two graphs G_1 and G_2 shown in Figure 6.

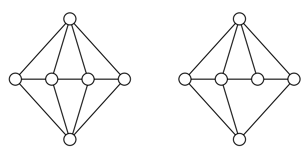

Figure 6: The non-EPT graphs G_1 and G_2.

That $\overline{P}_6$ is a non-EPT graph shows particularly that no antihole $\overline{C}_k$ for $k \geq 7$ can occur in such graphs. This implies:

Theorem 2 (Golumbic and Jamison [9]) *An EPT graph is perfect if and only if it does not contain an odd hole.*

Based on this result, we further examine which minimal non-comparability non-superperfect graphs can occur in edge intersection graphs of paths in a tree:

Theorem 3 *If $\mathcal{P}$ is a set of paths in a* tree*, then the EPT graph $I(\mathcal{P})$ can contain $\overline{A}_1$, $\overline{A}_2$ and*

- *odd holes C_{2k+1} for $k \geq 2$, but no odd antiholes $\overline{C}_{2k+1}$ for $k \geq 3$,*
- *the graphs J_k and J'_k for all $k \geq 2$,*
- *$\overline{D}_2$, $\overline{D}_3$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, $\overline{F}_1$, $\overline{F}_2$, $\overline{F}_3$, but none of $\overline{D}_k$, $\overline{E}_k$, $\overline{F}_k$ for $k \geq 4$.*

Proof (Sketch) In order to prove the theorem, we will present according path collections for the affirmative cases and exhibit the non-EPT graph $\overline{P}_6$ as common induced subgraph of the remaining cases.

If $\mathcal{P}$ is a set of paths in a tree, then $I(\mathcal{P})$ can contain

- $\overline{A}_1$ and $\overline{A}_2$ (see Fig. 7),
- odd holes C_{2k+1} for $k \geq 2$ by Theorem 1,
- the graphs J_k and J'_k for all $k \geq 2$ (see Fig. 8 for a representation of J_2 and J'_2 which can be easily extended to all cases for $k \geq 2$),
- the graphs $\overline{D}_2$, $\overline{D}_3$, $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, $\overline{F}_1$, $\overline{F}_2$, $\overline{F}_3$ (see Fig. 9, Fig. 10, Fig. 11 and Fig. 12).

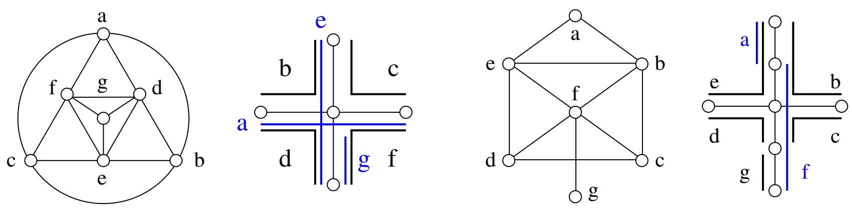

Figure 7: The graphs $\overline{A}_1 = I(\mathcal{P})$ and $\overline{A}_2 = I(\mathcal{P})$ with $\mathcal{P}$ in a tree.

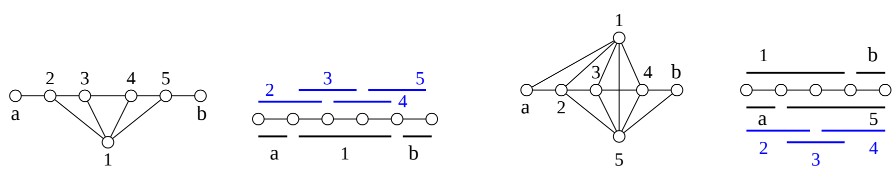

Figure 8: The graphs $J_2 = I(\mathcal{P})$ and $J'_2 = I(\mathcal{P})$ with $\mathcal{P}$ in a path.

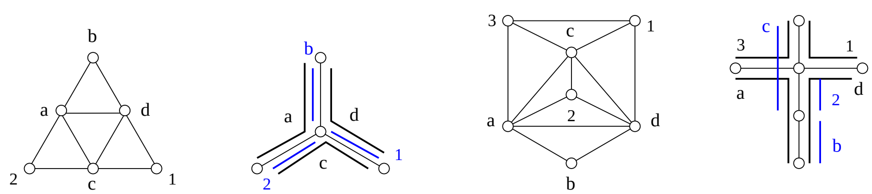

Figure 9: The graphs $\overline{D}_2 = I(\mathcal{P})$ and $\overline{D}_3 = I(\mathcal{P})$ with $\mathcal{P}$ in a tree.

That $\overline{P}_6$ is not an EPT graph by [9] clearly excludes odd antiholes $\overline{C}_{2k+1}$ for $k \geq 3$ (see Theorem 2). Note further that the graphs D_k, E_k, F_k contain a P_6 for all $k \geq 4$ from their definition, see Fig. 2. Thus, $\overline{D}_k$, $\overline{E}_k$, $\overline{F}_k$ have a $\overline{P}_6$ as induced subgraph, and cannot be EPT graphs. $\square$

This implies that perfect EPT graphs are not necessarily superperfect. We next examine the situation when we restrict the tree further. A graph is *triangulated* if it does not have holes C_k with $k \geq 4$ as induced subgraph. We can show the following:

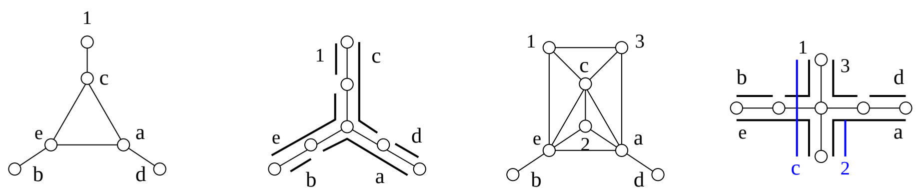

Figure 10: The graphs $\overline{E}_1 = I(\mathcal{P})$ and $\overline{E}_3 = I(\mathcal{P})$ with $\mathcal{P}$ in a tree.

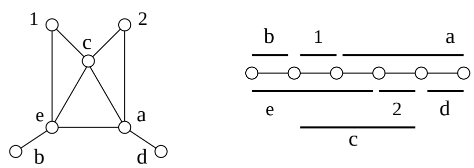

Figure 11: The graph $\overline{E}_2 = I(\mathcal{P})$ with $\mathcal{P}$ in a path.

Lemma 4 *If $\mathcal{P}$ is a set of paths in a* tree *with maximum degree 3, then $I(\mathcal{P})$ is* triangulated *and can contain J_k and J'_k for all $k \geq 2$ and $\overline{D}_2$, $\overline{E}_1$, $\overline{E}_2$.*

Proof (Sketch) For that, recall from Theorem 1 that C_k with $k \geq 4$ can occur in EPT graphs only if the tree contains a star $K_{1,k}$. This implies for the case where the network is a tree with maximum degree 3 that $I(\mathcal{P})$ cannot contain any hole C_k with $k \geq 4$, hence $I(\mathcal{P})$ is triangulated and from the affirmative cases of Theorem 3, we can exclude the following graphs:

- all odd holes C_{2k+1} for $k \geq 2$,
- the graphs $\overline{A}_1$, $\overline{A}_2$, $\overline{D}_3$, $\overline{E}_3$, $\overline{F}_1$, $\overline{F}_2$, $\overline{F}_3$ (that all contain a C_4),

whereas the graphs J_k and J'_k for all $k \geq 2$ and $\overline{D}_2$, $\overline{E}_1$, $\overline{E}_2$ have an according representation (see Fig. 8, ..., Fig. 11). □

This implies that edge intersection graphs of paths in a tree with maximum degree 3 are perfect (as they neither contain odd holes nor odd antiholes), but not necessarily superperfect.

Moreover, an *interval graph* is the intersection graph of intervals in a line.

Lemma 5 *If $\mathcal{P}$ is a set of paths in a* path, *then $I(\mathcal{P})$ is an* interval graph *and can contain the graphs J_k and J'_k for all $k \geq 2$ and $\overline{E}_2$.*

Proof In this case, $I(\mathcal{P})$ is clearly an interval graph (here represented as subpaths in a path) and from the affirmative cases of Lemma 4, only $\overline{E}_2$ and the graphs J_k and J'_k for all $k \geq 2$ remain (see their according representations in Fig. 8 and Fig. 11), whereas $\overline{D}_2$, $\overline{E}_1$ are excluded (as known examples of non-interval graphs from [14]). □

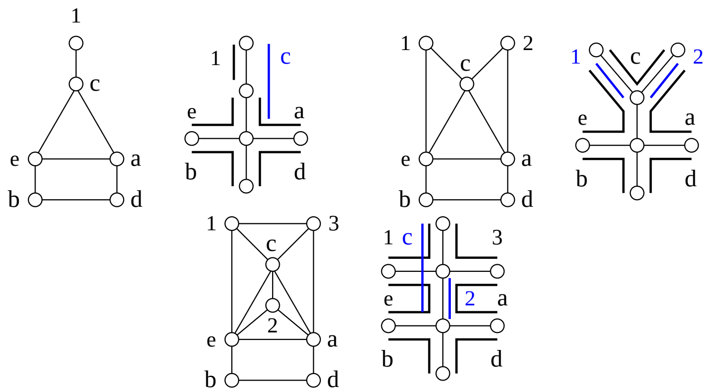

Figure 12: The graphs $\overline{F}_1 = I(\mathcal{P})$, $\overline{F}_2 = I(\mathcal{P})$ and $\overline{F}_3 = I(\mathcal{P})$ with $\mathcal{P}$ in a tree.

This implies that even edge intersection graphs of paths in a path are not necessarily superperfect.

We next briefly discuss which further minimal non-superperfect graphs can be EPT graphs. Recall that all of them have to be perfect and have to contain a minimal non-comparability superperfect graph as proper induced subgraph. Among the minimal non-comparability superperfect graphs, the following are EPT graphs:

- $\overline{C}_6$ (but no even antihole $\overline{C}_{2k}$ for $k \geq 4$ as they all contain $\overline{P}_6$),
- none of the graphs J''_k for all $k \geq 3$ (as they all contain G_1 induced by the nodes $1, 2, 3, 4, 5, 2k$),
- the graphs $\overline{A}_3, \ldots \overline{A}_6, \overline{A}_8, \ldots \overline{A}_{10}$ (but not $\overline{A}_7$ as it has a G_2).

Hence, any further example of a minimal non-superperfect EPT graph has to contain one of $\overline{C}_6$, $\overline{A}_3$, $\ldots \overline{A}_6$, $\overline{A}_8$, $\ldots \overline{A}_{10}$ as proper induced subgraph. Fig. 13 shows one example containing $\overline{A}_{10}$ together with a weight vector d causing a gap between weighted clique and interval chromatic number and a path representation as EPT graph.

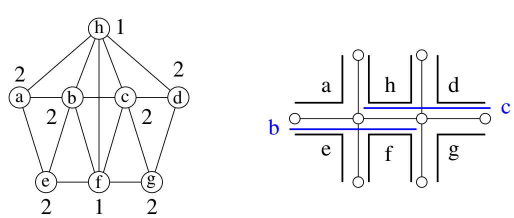

Figure 13: A minimal non-superperfect EPT graph containing $\overline{A}_{10}$.

3 If the network is a cycle

If the underlying communication network is a cycle C, then there exist exactly two (u,v)-paths P_{uv} in C for every traffic demand between a pair u,v of nodes. Hence, if C is a cycle, then the number of possible routings $\mathcal{P}$ (and their edge intersection graphs $I(\mathcal{P})$) is exponential in the number $|\mathcal{D}|$ of end-to-end traffic demands, namely $2^{|\mathcal{D}|}$.

Moreover, the edge intersection graphs of paths in a cycle are clearly *circular-arc graphs* (that are the intersection graphs of arcs in a cycle, here represented as paths in a hole C_n).

It is well-known that circular-arc graphs are not necessarily perfect as they can contain both odd holes and odd antiholes, see e.g. [5] and Fig. 14 for illustration.

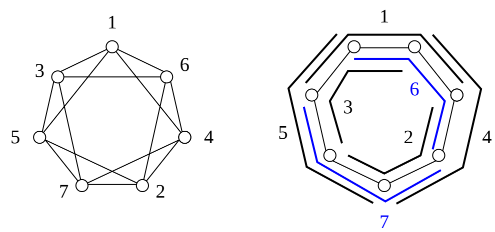

Figure 14: The odd antihole $\overline{C}_7 = I(\mathcal{P})$ with $\mathcal{P}$ in a cycle.

In order to address the question which of the studied perfect minimal non-comparability, non-superperfect graphs can occur in circular-arc graphs, we either present according path collections for the affirmative cases or exhibit a minimal non-circular-arc graph otherwise.

Theorem 6 *If $\mathcal{P}$ is a set of paths in a* cycle*, then the circular-arc graph $I(\mathcal{P})$ can contain $\overline{A}_1$ but not $\overline{A}_2$,*

- *all odd holes C_{2k+1} and odd antiholes $\overline{C}_{2k+1}$ for $k \geq 2$,*
- *the graphs J_k and J'_k for all $k \geq 2$,*
- *$\overline{D}_2$, $\overline{D}_3$, $\overline{D}_4$, but not the graphs $\overline{D}_k$ for $k \geq 5$,*
- *$\overline{E}_1$ and $\overline{E}_2$, but not the graphs $\overline{E}_k$ for $k \geq 3$,*
- *$\overline{F}_2$, but not $\overline{F}_1$ neither the graphs $\overline{F}_k$ for $k \geq 3$.*

Proof (Sketch) If $\mathcal{P}$ is a set of paths in a cycle, then $I(\mathcal{P})$ can contain odd holes C_{2k+1} and odd antiholes $\overline{C}_{2k+1}$ for $k \geq 2$ (as they are well-known examples of circular-arc graphs, see e.g. [5] and Fig. 14 for illustration), the graphs J_k and J'_k for all $k \geq 2$ and $\overline{E}_2$ (as they are, by Lemma 5, examples of interval graphs, which form by construction a subclass of circular-arc graphs, see Fig. 8 and Fig. 11 for illustration). For the remaining affirmative cases, we can show that the graphs $\overline{A}_1$, $\overline{D}_2$, $\overline{D}_3$, $\overline{D}_4$, $\overline{E}_1$ and $\overline{F}_2$ are circular-arc graphs. The corresponding

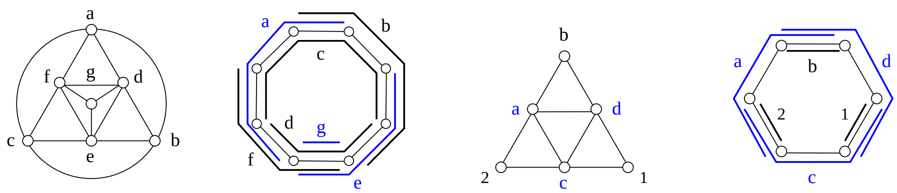

Figure 15: The graphs $\overline{A}_1 = I(\mathcal{P})$ and $\overline{D}_2 = I(\mathcal{P})$ with $\mathcal{P}$ in a cycle.

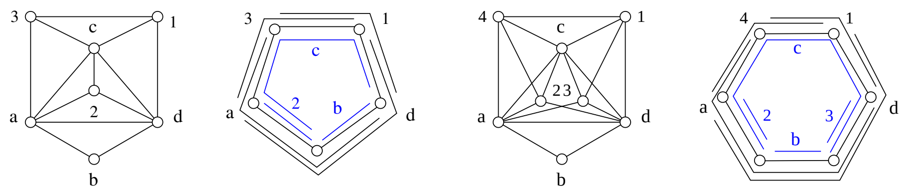

Figure 16: The graphs $\overline{D}_3 = I(\mathcal{P})$ and $\overline{D}_4 = I(\mathcal{P})$ with $\mathcal{P}$ in a cycle.

collections of paths are given in Fig. 15 for $\overline{A}_1$ and $\overline{D}_2$, Fig. 16 for $\overline{D}_3$ and $\overline{D}_4$, Fig. 17 for $\overline{E}_1$ and $\overline{F}_2$.

However, we can show by case analysis that $\overline{E}_3$ is not a circular-arc graph. $C_4 \cup K_1$ is a well-known minimal non-circular-arc graph by [2]. A $C_4 \cup K_1$ occurs as induced subgraph of

- $\overline{A}_2$ (induced by the nodes b, c, d, e and g, see Fig. 7),
- $\overline{F}_1$ (induced by the nodes 1 and a, b, d, e, see Fig. 12),
- each of $\overline{D}_k$ for $k \geq 5$ (induced by the nodes b and $1, 2, k-1, k$) and each of $\overline{E}_k$ for $k \geq 4$ (induced by the nodes b and $a, 1, k-1, k$), see Fig. 2 for the definitions.

The domino is another well-known minimal non-circular-arc graph by [5], and each of $\overline{F}_k$ for $k \geq 3$ contains a domino induced by $1, k, a, b, d, e$ as subgraph (see $\overline{F}_3$ in Fig. 12 for illustration). □

We next discuss which further minimal non-superperfect graphs can be circular-arc graphs. Recall that they have to be perfect and have to contain a minimal non-comparability superperfect proper induced subgraph. Among the perfect minimal non-comparability superperfect graphs, the following are circular-arc graphs:

- no even antihole $\overline{C}_{2k}$ for $k \geq 3$ ("folklore"),
- neither J''_3 (as it can be shown by case analysis to be not a circular-arc graph) nor the graphs J''_k for all $k \geq 4$ (as they all contain the well-known minimal non-circular-arc graph $K_{2,3}$ induced by the nodes $1, 2, 4, 6, 2k$),
- all of the graphs $\overline{A}_3, \ldots \overline{A}_{10}$.

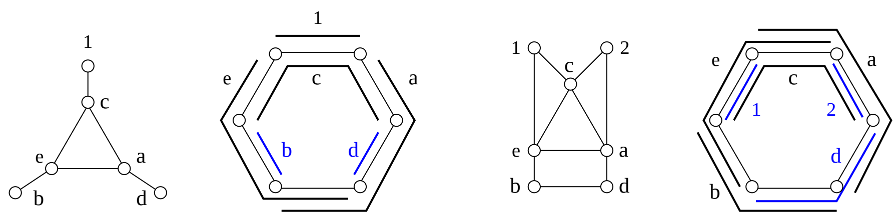

Figure 17: The graphs $\overline{E}_1 = I(\mathcal{P})$ and $\overline{F}_2 = I(\mathcal{P})$ with $\mathcal{P}$ in a cycle.

Remark 7 *Note that $\overline{E}_3$ and J_3'' are, to the best of our knowledge,* new *examples of minimal non-circular-arc graphs (see e.g. the results on circular-arc graphs surveyed in [5]).*

Hence, any further example of a minimal non-superperfect circular-arc graph has to contain one of $\overline{A}_3, \ldots \overline{A}_{10}$ as proper induced subgraph. Fig. 18 shows one example containing $\overline{A}_6$ together with a weight vector d causing a gap between weighted clique and interval chromatic number and a path representation as circular-arc graph.

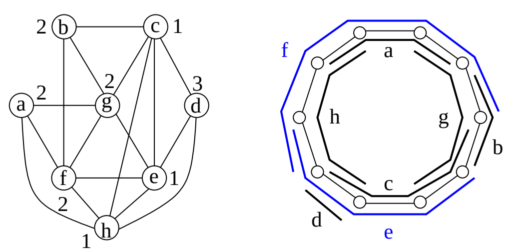

Figure 18: A minimal non-superperfect circular-arc graph containing $\overline{A}_6$.

4 The general case

Modern optical networks have clearly not a tree-like structure neither are just cycles due to survivability aspects concerning node or edge failures in the network G, see e.g. [13]. Instead, today's optical networks are 2-connected, sparse planar graphs with small maximum degree and have more a grid-like structure, see Fig. 19 showing a network of Spain taken from [15] as example.

We first wonder which minimal non-comparability non-superperfect graphs can occur in edge intersection graphs of paths in such networks G and can show:

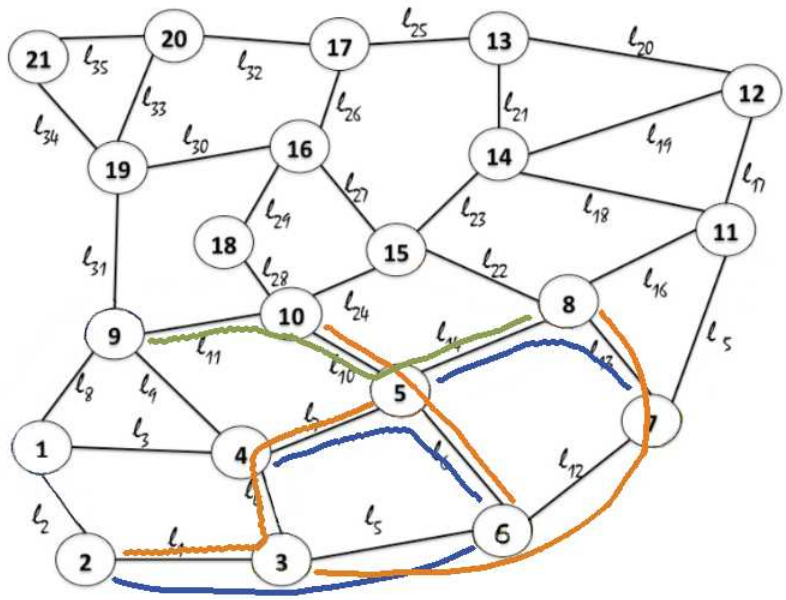

Figure 19: A network of Spain together with a routing $\mathcal{P}$ of a subset of demands with $I(\mathcal{P}) = C_7$.

Theorem 8 *All minimal non-comparability non-superperfect graphs can occur in edge intersection graphs $I(\mathcal{P})$ of sets $\mathcal{P}$ of paths in networks G.*

Proof (Sketch) In this case, all studied minimal non-superperfect graphs occurring in $I(\mathcal{P})$ when the network G is a tree or a cycle can clearly be present, hence we conclude from Theorem 3 and Theorem 6 that we have

- $\overline{A}_1$ and $\overline{A}_2$ (can occur when G is a tree),
- all odd holes C_{2k+1} for $k \geq 2$ (can occur in both cases),
- all odd antiholes $\overline{C}_{2k+1}$ for $k \geq 2$ (can occur when G is a cycle),
- the graphs J_k and J'_k for all $k \geq 2$ (can occur in both cases),
- the graphs $\overline{D}_1$, $\overline{D}_2$, $\overline{D}_3$ (can occur when G is a cycle),
- $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, $\overline{F}_1$, $\overline{F}_2$, $\overline{F}_3$ (can occur when G is a tree).

Hence, it is left to address the families $\overline{D}_k$ for $k \geq 5$ and $\overline{E}_k$, $\overline{F}_k$ for $k \geq 4$. We will present the corresponding collections of paths. For that, we first note that the graphs D_k, E_k, F_k contain a P_{k+2} for all $k \geq 4$ from their definition (see Fig. 2), induced by the nodes $a, 1, \ldots, k, d$ in $\overline{D}_k$ and by the nodes $a, 1, \ldots, k, e$ in $\overline{E}_k$, $\overline{F}_k$. Thus, $\overline{D}_k$, $\overline{E}_k$, $\overline{F}_k$ have a $\overline{P}_{k+2}$ which can be represented as circular-arc graph by using the corresponding paths of a path representation of a sufficiently large odd antihole in a cycle C.

In all three cases, the path $P(c)$ of node c needs to be large enough to share an edge with all paths corresponding to the nodes in this $\overline{P}_{k+2}$ (e.g. $P(c)$ can occupy all but one edge of the cycle C).

For $\overline{D}_k$, it is only left to place path $P(b)$ of node b which has to share an edge only with $P(a)$ and $P(d)$. For that, we add a node z and join it with two nodes x and y on C being endnodes of $P(a)$ and $P(d)$ on C. We extend $P(a)$ by the edge xz and $P(d)$ by the edge yz and let occupy $P(b)$ exactly these two edges xy and yz.

For $\overline{E}_k$ and $\overline{F}_k$, it is left to place path $P(b)$ of node b (which has to share an edge only with $P(e)$) and path $P(d)$ of node d (which has to share an edge only with $P(a)$). We add a node z and join it with two nodes x and y on C being endnodes of $P(a)$ and $P(e)$ on C. We extend $P(a)$ by the edge xz and $P(e)$ by the edge yz.

For $\overline{E}_k$, $P(b)$ and $P(d)$ must be edge-disjoint: we choose $P(b) = yz$ and $P(d) = xz$. For $\overline{F}_k$, $P(b)$ and $P(d)$ must share an edge: we add a further node z', join it with z, and let $P(b) = y, z, z'$ and $P(d) = x, z, z'$, see Fig. 20 for illustration.

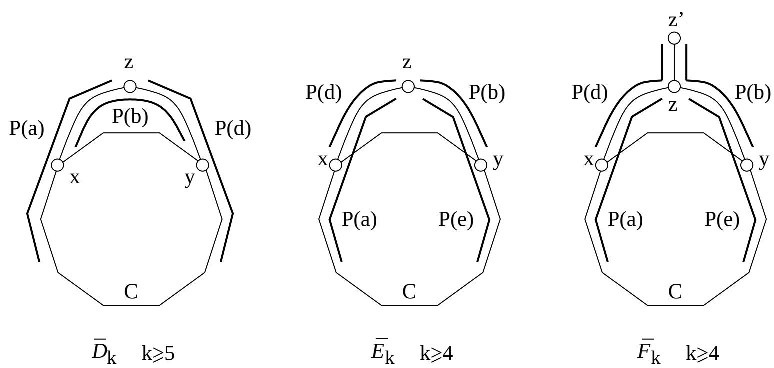

Figure 20: Illustration for the path representation of $\overline{D}_k$ for $k \geq 5$ and $\overline{E}_k$, $\overline{F}_k$ for $k \geq 4$ as edge intersection graphs of paths in networks.

Hence, the families $\overline{D}_k$ for $k \geq 5$ and $\overline{E}_k$, $\overline{F}_k$ for $k \geq 4$ have representations as edge intersection graphs of paths in a sparse planar graph. This finally proves the theorem. $\square$

In addition, there are further minimal non-superperfect graphs in edge intersection graphs of paths in networks. Fig. 21 shows two examples containing $\overline{C}_6$ (the smallest minimal non-comparability superperfect graph) together with a weight vector d causing a gap between weighted clique and interval chromatic number. Note that these two graphs are neither EPT graphs (as they contain a G_2 induced by the nodes $a, \ldots, g$) nor circular-arc graphs (as they contain a $\overline{C}_6$ induced by the nodes $a, \ldots, f$), but have a path representation in sparse planar graphs.

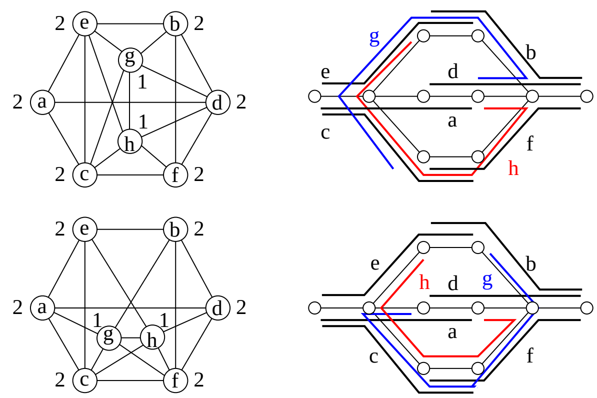

Figure 21: Minimal non-superperfect graphs containing $\overline{C}_6$ and their path representations in sparse planar graphs.

We expect that *all* minimal non-superperfect graphs can occur in edge intersection graphs of paths in networks, as soon as the networks G satisfy minimal survivability conditions concerning edge or node failures.

5 Concluding remarks

From the fact that both, EPT graphs and circular-arc graphs, are not necessarily perfect, we notice that also edge intersection graphs of paths in networks are not necessarily perfect and, thus, also not necessarily superperfect. If we restrict the networks to trees having maximum degree 3, then $I(\mathcal{P})$ is triangulated (and, thus, perfect), but not necessarily superperfect, and even if we restrict the networks to paths, then $I(\mathcal{P})$ is an interval graph, but still not necessarily superperfect (as the minimal non-superperfect graphs J_k and J'_k for all $k \geq 2$ and $\overline{E}_1$ can occur). This is in accordance with the fact that the SA problem has been showed to be NP-hard on paths [16].

Hence, in all networks, it depends on the weights d induced by the traffic demands whether there is a gap between the weighted clique number $\omega(I(\mathcal{P}), d)$ and the interval chromatic number $\chi_I(I(\mathcal{P}), d)$. To determine the size of this gap, we propose to extend the concept of χ-binding functions introduced in [11] for usual coloring to interval coloring in weighted graphs, that is, to χ_I-binding functions f with

$$\chi_I(I(\mathcal{P}), d) \leq f(\omega(I(\mathcal{P}), d))$$

for edge intersection graphs $I(\mathcal{P})$ in a certain class of networks and all possible non-negative integral weights d. We can identify for one of the studied families of minimal non-superperfect graphs such a χ_I-binding function:

Lemma 9 *If $I(\mathcal{P})$ is an odd hole, then $\chi_I(I(\mathcal{P}), d) \leq \frac{3}{2}\omega(I(\mathcal{P}), d)$ for all non-negative integral weights d.*

Note that a network of Spain together with its demands (taken from [15]) is a real instance where a natural routing $\mathcal{P}$ yields an edge intersection graph $I(\mathcal{P})$ with several non-superperfect subgraphs, including an odd hole C_7 that attains the worst-case bound of the χ_I-binding function.

It is clearly of interest to study such χ_I-binding functions for other families of minimal non-superperfect graphs and to identify a hierarchy of graph classes between trees respectively cycles and sparse planar graphs resembling the structure of modern optical networks in terms of the gap between $\omega_I(I(\mathcal{P}), d)$ and $\chi_I(I(\mathcal{P}), d)$.

Furthermore, in networks different from trees, the routing part of the RSA problem is crucial and raises the question whether it is possible to select the routes in $\mathcal{P}$ in such a way that neither non-superperfect graphs nor unnecessarily large weighted cliques occur in $I(\mathcal{P})$.

Finally, giving a complete list of minimal non-superperfect graphs is an open problem, so that our future work comprises to find more minimal non-superperfect graphs and to examine the here addressed questions for them.

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Appendix

We here discuss a network of Spain together with its demands as a real instance of the RSA problem (taken from [15]), see Fig. 19 for the network and Table 1 for a subset of demands.

index	origin s	destination t	demand d_{st}
1	6	2	3
2	3	8	3
3	5	7	3
4	8	9	3
5	10	6	3
6	4	6	3
7	2	5	3

Table 1: The considered subset of demands

A natural routing $\mathcal{P}$ of this subset of demands (see again Fig. 19) yields an edge-intersection graph $I(\mathcal{P})$ equal to an odd hole C_7:

- $P_{6,2}$ and $P_{3,8}$ share link l_5,
- $P_{3,8}$ and $P_{5,7}$ share link l_{13},
- $P_{5,7}$ and $P_{8,9}$ share link l_{14},
- $P_{8,9}$ and $P_{10,6}$ share link l_{10},
- $P_{10,6}$ and $P_{4,6}$ share link l_6,
- $P_{4,6}$ and $P_{2,5}$ share link l_7,
- $P_{2,5}$ and $P_{6,2}$ share link l_1.

The odd hole C_7 together with the weights $d = (3, \ldots, 3)$ attains the worst-case bound of the χ_I-binding function as it has $\omega(C_7, d) = 6 < 9 = \chi_I(C_7, d)$.